POLYNOMIAL FUNCTIONS ON $O_{2\ell+1}$

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To Hilary

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Summary.

This thesis is an attempt to generalise to the odd orthogonal group Γ_K, over an infinite field K not of characteristic two, the work of Schur [S], and more recently Green [G], on the general linear group G_K using the approach of Weyl [W] in characteristic zero. The special feature here is that we treat Γ_K as merely a group of matrices defined by the vanishing of polynomials in its coefficients (the classical view) rather than a group generated by elements derived from an associated Lie algebra, the approach used initially by Chevalley and adopted by most authors in recent times.

After generalising Green's [G] Schur algebra for G_K to Γ_K in §0 we prove in §1 Chevalley's famous theorem on the 'Big Cell' in G_K and then, by an easy extension, prove it for the Big Cell in Γ_K. Chevalley's original proof uses representations of Lie algebras, ours requires nothing but a little knowledge of the coordinate ring $K_+[G]$ of all 'polynomial' functions on G_K. We define $K[\Gamma]$, the coordinate ring of Γ_K, to be the space of all polynomial functions on G_K restricted to Γ_K and in §2 give a generating set of the kernel of the restriction map $\Psi_K: K_+[G] \to K[\Gamma]$. This generalises Weyl's result in characteristic zero. In §3 we use this result to show that the family, or 'scheme', of rings $K[\Gamma]$ (K varying over all infinite fields not of characteristic two) is 'defined over Z'; in fact $K[\Gamma]$ is naturally isomorphic to $K \otimes Z[\Gamma_Q]$, where $Z[\Gamma_Q]$ is the subring of $Q[\Gamma]$ spanned by 'monomial' functions. This enables us to formulate a 'modular' representation theory for Γ which connects polynomial representations of Γ_Q with those of Γ_K.

In §4 we investigate the Schur algebras of Γ_Q following Weyl [W] and in §5 find a complete set of irreducibles for each of them, once again following the lead of Weyl. In §6 we attempt to 'reduce' these modules modulo p to obtain 'Weyl' modules for Γ_K, a task only partially completed.

§0. Preliminaries and notation

0.0

Throughout K will denote an infinite field of arbitrary characteristic unless stated otherwise and, when no confusion should arise, we shall write the tensor product $\otimes_K$ as $\otimes$. For an integer $n > 0$, $E_K(n)$ (or just E_K) will denote an n-dimensional K-space with basis $\{e_1, e_2, \ldots, e_n\}$. If $r > 0$ is another integer we denote the r-fold tensor product $E_K \otimes E_K \otimes \ldots \otimes E_K$ by E_K^r and define E_K^0 to be K.

Denote by $I(n,r)$ the set of r-tuples with entries from $\underline{n} = \{1,2,\ldots,n\}$. Then E_K^r has basis

$$\{e_i := e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_r} : i = (i_1, i_2, \ldots, i_r) \in I(n,r)\} .$$

0.1 The general linear group and polynomial functions

We denote the group of all non-singular $n \times n$ matrices $g = (g_{\mu\nu})_{\mu,\nu \in \underline{n}}$ with entries in K by $G_{n,K}$ or, when no confusion should arise, variously by G_n, G_K, $G(K)$ and G.

Then $G_{n,K}$ acts naturally on the left of $E_K(n)$ by extending linearly to the whole of $E_K(n)$ the action

0.1a

$$g.e_\nu = \sum_{\mu \in \underline{n}} g_{\mu\nu} e_\mu \quad \text{for all } \nu \in \underline{n},\ g \in G_{n,K} .$$

Hence $G_{n,K}$ acts on $E_K^r(n)$ by extending linearly to the whole of $E_K^r(n)$ $(r > 0)$ the action

0.1b

$$g(e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_r}) = g.e_{i_1} \otimes g.e_{i_2} \otimes \ldots \otimes g.e_{i_r}$$

(for all $(i_1, i_2, \ldots, i_r) \in I(n,r)$, $g \in G_{n,K}$) and $G_{n,K}$ acts trivially on $E_K^0(n) = K$.

Thus, the group algebra $KG_{n,K}$ of $G_{n,K}$ consisting of formal sums

$$\sum_{g \in G_{n,K}} \lambda_g . g \qquad (\lambda_g \in K, \text{ finitely many } \neq 0)$$

acts on $E_K^r(n)$, making it into a left $KG_{n,K}$-module.

0.1c

For any set Ω we define K^Ω to be the set of functions $f : \Omega \to K$. Then K^Ω is a commutative K-algebra with operations defined 'pointwise' e.g. if $f, f' \in K$ then $ff' \in K^\Omega$ is the function which takes $x \in \Omega$ to $f(x)f'(x) \in K$. Let Γ be any group, then we can extend each $f \in K^\Gamma$ linearly to the whole of the group algebra $K\Gamma$ of Γ and identify K^Γ with $\mathrm{Hom}_K(K\Gamma, K)$, the K-algebra of K-linear maps $K\Gamma \to K$.

Now for $\mu, \nu \in \underline{n}$ define $c_{\mu\nu} \in K^{G_{n,K}}$ to be the K-linear function which maps each $g \in G_{n,K}$ to its $(\mu,\nu)^{th}$ coefficient $g_{\mu,\nu} \in K$. Denote by $K_+[G_{n,K}]$ the K-subalgebra of $K^{G_{n,K}}$ generated by these functions. Then $K_+[G_{n,K}]$ is by definition the K-algebra of polynomial functions on $G_{n,K}$ and since K is infinite the $c_{\mu\nu}$ $(\mu, \nu \in \underline{n})$ are algebraically independent over K . Thus $K_+[G_{n,K}]$ can be regarded as the K-algebra of polynomials in n^2 indeterminates $c_{\mu\nu}$ $(\mu,\nu \in \underline{n})$. For an integer $r \geq 0$ we denote by $K^r[G_{n,K}]$ the K-subspace of $K_+[G_{n,K}]$ spanned by monomials in the $c_{\mu\nu}$ $(\mu,\nu \in \underline{n})$ which are of degree r , (thus $K^0[G_{n,K}] = K\mathbb{1}$ where $\mathbb{1}$ is the constant function $g \to 1_K$ $(g \in G_{n,K})$). Thus the dimension of $K^r[G_{n,K}]$ as a K-space is $\binom{n^2+r-1}{r}$.

For $r > 0$, let $G(r)$ be the symmetric group on $\underline{r} = \{1,2,\ldots,r\}$. Then $G(r)$ acts on the right of $I(n,r)$ by

0.1d

$$(i_1,i_2,\ldots,i_r).\pi = (i_{\pi(1)},i_{\pi(2)},\ldots,i_{\pi(r)})$$

(for all $(i_1,i_2,\ldots,i_r) \in I(n,r)$, $\pi \in G(r)$) .

Hence $G(r)$ acts on $I(n,r) \times I(n,r)$: if $i,j \in I(n,r)$ and $\pi \in G(r)$ then $(i,j)\pi = (i\pi,j\pi)$.

For $i,j \in I(n,r)$ we write $i \sim j$ if i and j are in the same $G(r)$ orbit of $I(n,r)$, that is if $i = j\pi$ for some $\pi \in G(r)$. Similarly for $h,k \in I(n,r)$ we write $(i,h) \sim (j,k)$ if $i = j\pi$ and $h = k\pi$ for some $\pi \in G(r)$. Denote by $T(n,r)$ a set of representatives of the $G(r)$ orbits in $I(n,r) \times I(n,r)$.

Now, given $i,j \in I(n,r)$ we write c_{ij} for the element $c_{i_1j_1}c_{i_2j_2}\ldots c_{i_rj_r}$ in $K^r[G_{n,K}]$.

Then

0.1e

$$K^r[G_n] = \bigoplus_{(i,j)\in T(n,r)} K.c_{ij}$$

since clearly c_{ij} runs over all monomials of degree r in the n^2 generators $c_{\mu\nu}$ $(\mu,\nu \in \underline{n})$ of $K_+[G_n]$, as (i,j) runs over $T(n,r)$. Hence

0.1f

$$|T(n,r)| = \dim_K K^r[G_n] = \binom{n^2+r-1}{r} \quad .$$

Let V be a finite dimensional left KG_n-module with basis $\{v_1, v_2, \ldots, v_m\}$. Then

0.1g

$$g.v_b = \sum_{a\in\underline{m}} p_{ab}(g)v_a \qquad \text{for all } b\in\underline{m},\ g\in G_n$$

where the $p_{ab} \in K^{G_n}$ $(a,b\in\underline{m})$, (called 'coefficient functions' of V).

Denote by $cf(V)$ the K-subspace of K^{G_n} spanned by the p_{ab} $(a,b\in\underline{m})$. It is easy to show that this definition is independent of the choice of basis of V. We call it the coefficient space of V.

We denote by $M_K(G_n)$ the category of finite dimensional left KG_n-modules V such that $cf(V) \subset K_+[G_n]$. Then every V in $M_K(G_n)$ gives rise to a finite dimensional representation $\rho_V : G_n \to GL(V)$ of G_n called a polynomial representation. This extends linearly to a representation (also denoted by ρ_V) of KG_n, $\rho_V : KG_n \to \mathrm{End}_K(V)$.

0.1h Example

The module E_K^r is in the category $M_K(G_n)$ for all $r \geq 0$, since by (0.1a,b) we have equations

$$g.e_j = \sum_{i \in I(n,r)} c_{ij}(g)e_i \quad \forall\ j \in I(n,r),\ g \in G_{n,K} .$$

Hence $cf(E_K^r) = K\text{-span } \{c_{ij} : i,j \in I(n,r)\}$.

Let $M_K^r(G_n)$ denote the subcategory of $M_K(G_n)$ consisting of those modules $V \in M_K(G_n)$ such that $cf(V) \subseteq K^r[G_n]$. These are the polynomial modules which afford representations whose coefficients are homogeneous of degree r in the $c_{\mu\nu}$ $(\mu,\nu \in \underline{n})$. Clearly E_K^r is in $M_K^r(G_n)$.

0.1i Theorem

Let $V \in M_K(G_n)$, then

$$V = \sum_{r \geq 0}^{\oplus} V^{(r)}$$

where each $V^{(r)}$ is a KG_n-submodule of V with $V^{(r)} \in M_K^r(G_n)$.

Proof

The proof given by Schur [S,p.5] for $K = \mathbb{C}$, the complex numbers, works for any infinite field.

By the theorem above, to study the polynomial representations of $G_{n,K}$

it is enough to study homogeneous representations ie. those in $M_K^r(G_n)$ for all $r \geq 0$.

Remark

$K_+[G_n]$ can be regarded as the coordinate ring of the affine semigroup $M_{n,K}$ of all $n \times n$ matrices with entries in K . This means we can consider polynomial representations of $G_{n,K}$ as rational representations of $M_{n,K}$ and vice versa.

Finally, we write

0.1j

$$K_r[G_n] := \sum_{0 \leq s \leq r}^{\oplus} K^r[G_n]$$

the space of polynomials in the $c_{\mu\nu}$ $(\mu,\nu \in \underline{n})$ of degree $\leq r$.

0.2 The Schur algebra

Let C be a K-coalgebra with comultiplication $\Delta: C \to C \otimes C$ and counit $e: C \to K$, (see [Sw] for definition of coalgebra and related terminology).

0.2a Examples

(1) We can consider K as a one dimensional K-coalgebra with $\Delta(1_K) = 1_K \otimes 1_K$ and $e(1_K) = 1_K$.

(2) The K-algebra $K_+[G]$ has a K-coalgebra structure. Comultiplication is defined by extending linearly and multiplicatively to the whole of $K_+[G]$: $\Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$ and

0.2b

$$\Delta(c_{\mu\nu}) = \sum_{\rho\in\underline{n}} c_{\mu\rho} \otimes c_{\rho\nu} \qquad (\mu,\nu \in \underline{n}) \quad .$$

This arises from the equations

$$c_{\mu\nu}(g.g') = \sum_{\rho\in\underline{n}} c_{\mu\rho}(g)c_{\rho\nu}(g') \quad (\mu,\nu \in \underline{n})$$

where $g,g' \in G_{n,K}$.

The counit $e:K_+[G] \to K$ is defined by extending $e(\mathbb{1}) = 1_K$ and

0.2c

$$e(c_{\mu\nu}) = \begin{cases} 1 & \mu = \nu \quad (\mu,\nu \in \underline{n}) \\ 0 & \mu \neq \nu \end{cases} .$$

This is evaluation at the identity of $G_{n,K}$.

(3) Since $\Delta(K^r[G]) \subseteq K^r[G] \otimes K^r[G]$ and $\Delta(K_r[G]) \subseteq K_r[G] \otimes K_r[G]$, both $K^r[G]$ and $K_r[G]$ are K-subcoalgebras of $K_+[G]$.

0.2d

We can make $C^* = \mathrm{Hom}_K(C,K)$ into an associative K-algebra with unit e by defining multiplication as follows:

if $f,f' \in C^*$ then $ff' = (f \otimes f')\Delta:C \to K$ (where $K \otimes K$ is identified with K).

For any $r \geq 0$ we call the dual space $S_K^r(G_n)$ of $K^r[G_n]$ a Schur algebra following Green [G,p.11]. Then by (0.2d) if $r > 0$, $i,j \in I(n,r)$ and $\phi,\phi' \in S_K^r(G_n)$ we have

$$\phi.\phi'(c_{ij}) = \sum_{k \in I(n,r)} \phi(c_{ik})\phi'(c_{kj}) \quad .$$

We denote by ζ_{ij} the element of $S_K^r(G_n)$ defined by

0.2e

$$\zeta_{ij}(c_{hk}) = \begin{cases} 1 & (i,j) \sim (h,k) \\ 0 & (i,j) \not\sim (h,k) \end{cases} \quad (h,k \in I(n,r)) \quad .$$

Clearly $\zeta_{ij} = \zeta_{h,k}$ iff $(i,j) \sim (h,k)$ and therefore $S_K^r(G_n)$ has basis $\{\zeta_{ij} : (i,j) \in T(n,r)\}$. For any $r \geq 0$ define $\varepsilon : KG_n \to S_K^r(G_n)$ to be the unique K-linear map such that for $g \in G_{n,K}$, $\varepsilon(g)$ is evaluation at g , i.e.

$$\varepsilon(g)(c) = c(g) \qquad \text{for all} \quad c \in K^r[G_n] \quad .$$

We write ε_g for $\varepsilon(g)$; $\quad \varepsilon(1_G)$ is the counit of $K^r[G_n]$.

Let $\mathrm{mod}(S_K^r(G_n))$ denote the category of finite dimensional left $S_K^r(G_n)$ modules.

0.2f Proposition

For any $r \geq 0$ the K-linear map $\varepsilon : KG_n \to S_K^r(G_n)$ is an epimorphism of K-algebras and induces an equivalence of categories between $M_K^r(G_n)$

and $\mathrm{mod}(S_K^r(G_n))$ using the rule

$$g.v = \varepsilon_g . v \qquad \text{for all} \quad g \in G_{n,K}, v \in V$$

where $V \in M_K^r(G_n)$ or $\mathrm{mod}(S_K^r(G_n))$.

Proof

The case $K = \mathbb{C}$ is due to Schur [S], for the general case see Green [G, §2.4].

0.2g Remark

If $V \in M_K^r(G_n)$ has basis $\{v_1, v_2, \ldots, v_m\}$ such that for $b \in \underline{m}$, $g \in G_{n,K}$

$$g.v_b = \sum_{a \in \underline{m}} p_{ab}(g) v_a$$

then the action of $S_K^r(G_n)$ on V is given by

$$\phi . v_b = \sum_{a \in \underline{m}} \phi(p_{ab}) v_a \quad \text{for all} \quad b \in \underline{m} \, , \; \phi \in S_K^r(G_n) \; .$$

Define $S_{r,K}(G_n)$ to be the external direct sum

0.2h

$$S_K^r(G_n) \dot{\oplus} S_K^{r-1}(G_n) \dot{\oplus} \ldots \dot{\oplus} S_K^0(G_n)$$

and identify it with the K-algebra $\mathrm{Hom}_K(K_r[G_n], K)$. It is clear that

$S_{r,K}(G_n)$ has basis

0.2i

$$\{\zeta_{ij} : (i,j) \in T_r(n) : = T(n,r) \cup T(n,r-1) \cup \ldots \cup T(n,0)\} .$$

If $\phi = \sum_{(i,j)\in T_r(n)} a_{ij}\zeta_{ij} \in S_{r,K}(G_n)$ we denote by $\phi^{(\rho)}$ the component of ϕ in $S_K^{\rho}(G_n)$ ie.

$$\phi^{(\rho)} : = \sum_{(i,j)\in T(\rho,n)} a_{ij}\ \zeta_{ij} .$$

We shall sometimes write ϕ as the $r+1$-tuple $(\phi^{(r)},\phi^{(r-1)},\ldots,\phi^{(0)})$.

Let $\epsilon_r : KG_n \to S_{r,K}(G_n)$ be the unique K-linear map such that for $g \in G_{n,K}, \epsilon_r(g)$ is evaluation at g .

0.2j Proposition

For any $r \geq 0$ $\epsilon_r : KG_n \to S_{r,K}(G_n)$ is an epimorphism of K-algebras.

Proof

The case $r = 0$ is immediate. Clearly $\epsilon_r(KG_n) \subseteq S_{r,K}(G_n)$. Suppose $S_{r,K}(G_n) \not\subseteq \epsilon_r(KG_n)$, then there exists a non zero element $c \in K_r[G_n]$ such that $\epsilon_r(g)(c) = c(g) = 0$ for all $g \in G_{n,K}$. By the irrelevance of algebraic inequalities, [W,p.4] we have $c \equiv 0$, a contradiction.

0.3 Polynomial functions on subgroups of G.

Let Γ be any subgroup of G_K. Elements of $K_+[G]$ can be regarded, by restriction to Γ, as functions on Γ or linear functions on the group algebra $K\Gamma$. We define $K[\Gamma] \subset K^\Gamma$ to be the set of these restricted functions. Then $K[\Gamma]$ is a K-subalgebra of K^Γ and inherits a K-coalgebra structure from $K_+[G]$. The restriction map $\Psi_K : K_+[G] \to K[\Gamma]$ is a surjective morphism of K-algebras and K-coalgebras. The subspace $\Psi_K(K_r[G])$ inherits a K-coalgebra structure from $K_r[G]$, denote it by $K_r[\Gamma]$ and for $i,j \in I(n,r)$ write c^Γ_{ij} for $\Psi_K(c_{ij})$. Denote by $S_{r,K}(\Gamma)$ the dual of $K_r[\Gamma]$. By (0.2d) we can give $S_{r,K}(\Gamma)$ a K-algebra structure. The surjective morphism of K-coalgebras

$$\begin{aligned} \Psi_{r,K} : K_r[G] &\to K_r[\Gamma] \\ c_{ij} &\to c^\Gamma_{ij} \end{aligned}$$

induces an injective morphism of K-algebras

0.3a

$$S_{r,K}(\Gamma) \to S_{r,K}(G) \quad .$$

We shall identify $S_{r,K}(\Gamma)$ with its image in $S_{r,K}(G)$.

We call $S_{r,K}(\Gamma)$ a ' Schur algebra' of Γ.

0.3b Lemma

For any subgroup Γ of G and non-negative integer r we have

$$\epsilon_r(K\Gamma) = S_{r,K}(\Gamma)$$

where $K\Gamma$ is considered as a K-subalgebra of KG .

<u>Proof</u>

Let $\phi \in S_{r,K}(G)$, then $\phi \in S_{r,K}(\Gamma)$ iff the kernel of $\phi : K_r[G] \to K$ contains the kernel of $\Psi_{r,K}$. (then ϕ can be considered as a function on $K_r[\Gamma]$) .

Let $\phi = \varepsilon_r(a)$ $(a \in K\Gamma)$ then $c \in \ker \Psi_{r,K}$ implies that $c(a) = \varepsilon_r(a)(c) = 0$ and hence $\varepsilon_r(K\Gamma) \subsetneq S_{r,K}(\Gamma)$.

Now, suppose $S_{r,K}(\Gamma) \not\subset \varepsilon_r(K\Gamma)$. Then there exists some non-zero $c \in K_r[\Gamma]$ with $c = \Psi_{r,K}(c')$ for some $c' \in K_r[G]$ such that $\varepsilon_r(a)(c') = c'(a) = 0$ for all $a \in K\Gamma$ ie. $c' \in \ker\Psi_{r,K}$ so that $\Psi_{r,K}(c') = c = 0$, a contradiction.

We define $\varepsilon_r^\Gamma : K\Gamma \to S_{r,K}(\Gamma)$ to be the restriction of $\varepsilon_r : KG \to S_{r,K}(G)$ to $K\Gamma$.

Let V be a finite dimensional left $K\Gamma$-module with basis $\{v_1, v_2, \ldots, v_m\}$, then for $b \in \underline{m}$

$$g.v_b = \sum_{a \in \underline{m}} p^\Gamma_{ab}(g)v_a \qquad (g \in \Gamma)$$

where $p^\Gamma_{ab} \in K^\Gamma$ $\quad (a,b \in \underline{m})$.

As in §0.1 we define the coefficient space $cf(V)$ to be the K-span of the p^Γ_{ab} $(a,b \in \underline{m})$ and as before it is independent of the choice of basis.

Denote by $M_{r,K}(\Gamma)$ the category of finite dimensional left $K\Gamma$-modules

V such that $cf(V) \subset K_r[\Gamma]$. As an example a module in $M_{r,K}(G)$ is the direct sum of modules from $M_K^r(G)$, $M_K^{r-1}(G), \ldots, M_K^0(G)$ by (0.11).

We have the analogue of (0.2f).

0.3c Proposition

Let Γ be a subgroup of G . Then the category of finite dimensional left $S_{r,K}(\Gamma)$ modules and $M_{r,K}(\Gamma)$ are equivalent using the rule

$$a.v = \varepsilon_r^{\Gamma}(a).v \qquad (a \in K\Gamma,\ v \in V)$$

where V is in either of the categories.

Proof

The proof follows that of (0.2f) given in [G, §2.4] and uses (0.3b).

0.3d Remark

Since, for $V \in M_K(\Gamma)$ we have $V \in M_{r,K}(\Gamma)$ for some $r \geq 0$, to study the $K\Gamma$ modules in $M_K(\Gamma)$ it is enough by (0.3c) to work with the finite dimensional associative K-algebras $S_{r,K}(\Gamma)$ $(r \geq 0)$. All this applies to $\Gamma = G$ of course, although by (0.11) and (0.2f) we can confine our attention to $S_K^\rho(G)$ $(\rho \geq 0)$.

In the case characteristic K equals zero, this was the technique used by Schur [S] on G and Weyl [W] on the other classical groups (see below). Green [G] has extended Schur's work to an arbitrary infinite field and we attempt here to extend Weyl's work.

0.4 Bilinear forms and subgroups of G

Let $B: E_K \times E_K \to K$ be a bilinear form defined by a matrix $\underline{B} = (B(e_\mu, e_\nu))_{\mu,\nu \in \underline{n}}$. Then the set

0.4a

$$\Gamma_B = \{g \in G_{n,K} \mid B(ge,ge') = B(e,e') \quad \forall\ e,e' \in E_K\}$$

is a subgroup of $G_{n,K}$ and if we denote by g^t the transpose of the matrix g , then

0.4b

$$\Gamma_B = \{g \in G_{n,K} \mid g^t \underline{B} g = \underline{B}\} \ .$$

0.4c Examples

For an integer $\ell > 0$, let

$$J_\ell = \begin{vmatrix} 0 & \cdots & 0 & 1_K \\ \vdots & & 1_K & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & & \vdots \\ 0 \quad 1_K & & & \vdots \\ 1_K \quad 0 & \cdots & & 0 \end{vmatrix} \qquad (\ell \times \ell)\text{-matrix.}$$

(i) The matrix $\underline{B} = \begin{vmatrix} 0 & J_\ell \\ -J_\ell & 0 \end{vmatrix}$ yields the symplectic group, $Sp_{2\ell}(K)$.

(ii) The matrix $\underline{B} = J_{2\ell}$ yields the even orthogonal group, $O_{2\ell}(K)$.

(iii) The matrix $\underline{B} = J_{2\ell+1}$ yields the odd orthogonal group, $O_{2\ell+1}(K)$.

Henceforth $\Gamma_{n,K}$ $(n = 2\ell+1)$, or when no confusion should arise Γ_K , $\Gamma(K)$ or Γ , will denote the odd orthogonal group $O_{2\ell+1}(K)$.

0.4d Remark

Many of the techniques used on $O_{2\ell+1}(K)$ in subsequent sections can also be applied, with some modifications, to the symplectic and even orthogonal groups. These similarities (and dissimilarities) are outlined in appendix B.

§1. Chevalley's theorem for G and $O_{2\ell+1}$

1.1 Unipotent and diagonal subgroups of G

We define the following subgroups of G :

U_G: = {unipotent upper triangular matrices}, a unipotent group

U_G^-: = {unipotent lower triangular matrices}, a unipotent group

T_G: = {non singular diagonal matrices}, a commutative group

W_G: = {permutation matrices}, isomorphic to $G(n)$.

1.1a

It is easy to see that $J_n U_{G_n} J_n = U_{G_n}^-$, where J_n is the matrix defined in (0.4c).

We can now state the following well known theorem.

1.1b Theorem (Bruhat, Chevalley)

We have $G = \dot{\bigcup\limits_{w \in W_G}} U_G^-.w.T_G.U_G$ (a disjoint union).

Proof

Steinberg [St, p.36] proves (by hand) that $G = \dot{\bigcup\limits_{w \in W_G}} U_G.w.T_G.U_G$.
Since J (= J_n) is a permutation matrix we can replace U_G by $U_G J$, also $JG = G$ and therefore

$$G = \dot{\bigcup_{w \in W_G}} (JU_GJ).w.\ T_G.U_G = \dot{\bigcup_{w \in W_G}} U_G^-.w.T_G.U_G.$$

1.1c

It is clear that $K[U_G]$ and $K[U_G^-]$ are polynomial rings over K freely generated by the $\binom{n}{2}$ coordinate functions $\{c_{\mu\nu}^U : 1 \leq \mu < \nu \leq n\}$ and $\{c_{\mu\nu}^{U^-} : 1 \leq \nu < \mu \leq n\}$ respectively, where $c_{\mu\nu}^U$ and $c_{\mu\nu}^{U^-}$ denote the image of $c_{\mu\nu} \in K_+[G]$ in $K[U_G]$ and $K[U_G^-]$ respectively. Also $K[T_G]$ is a polynomial ring over K freely generated by $\{c_{\mu\mu}^T : \mu \in \underline{n}\}$, where $c_{\mu\mu}^T$ is the image of $c_{\mu\mu} \in K_+[G]$ in $K[T_G]$. Clearly $K[T_G]$ has a basis of monomials:

1.1d

$$(c_{11}^T)^{\lambda_1}(c_{22}^T)^{\lambda_2} \cdots (c_{nn}^T)^{\lambda_n}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in Z_{\geq 0}$

Denote by $\Lambda(G)$ the set of n-tuples with entries from $Z_{\geq 0}$ and for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda(G)$ by $\chi_{\lambda,K}$ the function of T_G given by (1.1d) ie.

1.1e

$$\chi_{\lambda,K}(\mathrm{diag}(t_1, t_2, \ldots, t_n)) = t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_n^{\lambda_n},$$

(where $\mathrm{diag}(t_1, t_2, \ldots, t_n) \in T_G$ is the matrix with $t_1, t_2, \ldots, t_n$ on the diagonal from left to right).

We call elements of $\Lambda(G)$ 'weights' and for $\lambda \in \Lambda(G)$ $\chi_{\lambda,K} \in K^{T_G}$ is the 'character of weight λ'. Clearly $\{\chi_{\lambda,K} : \lambda \in \Lambda(G)\}$ is a basis of $K[T_G]$ with $\chi_{\lambda,K}$ corresponding to the monomial (1.1d).

1.2 The Big Cell and Chevalley's theorem for G

Define the Big Cell Ω_G in G to be the set of elements

1.2a

$\{u^- tu : u^- \epsilon U_G^-, t \in T_G, u \in U_G\}$

Now, for $g \in G_n$ denote by g^o the matrix obtained from g by deleting the n^{th} row and column. If $f \in K_+[G_n]$ is a polynomial in the $c_{\mu\nu}$ $(1 \leq \mu, \nu \leq n-1)$ we can consider f as an element of $K_+[G_{n-1}]$ by using the obvious inclusion $K_+[G_{n-1}] \subset K_+[G_n]$. (recall $K_+[G_{n-1}]$ is free on the indeterminates $\{c_{\mu\nu} : 1 \leq \mu,\nu \leq n-1\}$ and $K_+[G_n]$ on indeterminates $\{c_{\mu\nu} : 1 \leq \mu,\nu \leq n\}$). Then, $f(g) = f(g^o)$, remembering that if g^o is singular we can still define $f(g^o)$ (Remark, p.6). It is easy to prove that:

1.2b

For $u^- \epsilon U_{G_n}^-$, $w \in W_{G_n}$, $t \in T_{G_n}$ and $u \in U_{G_n}$ then $(u^- wtu)^o = u^{-o} w^o t^o u^o$. Also $\Omega_{G_n}^o = \Omega_{G_{n-1}}$.

We now prove the following important result.

1.2c Proposition

For $s \in \underline{n}$ define $D_s \in K_+[G_s] \subset K_+[G_n]$ to be the determinant of the matrix $(c_{\mu\nu})_{\mu,\nu \in \underline{s}}$ and $d_n := \prod_{s \in \underline{n}} D_s$.

Then $\Omega_{G_n} = \{g \in G_n \mid d_n(g) \neq 0\}$.

Proof

We prove this by induction on n. Clearly the result holds if $n = 1$. Assume the result is true for $n-1$, so that $\Omega_{G_{n-1}} = \{g \in G_{n-1} | d_{n-1}(g) \neq 0\}$. Consider $u^{-}tu \in U^{-}_{G_n} T_{G_n} U_{G_n}$, then by (1.2b) $u^{-0}t^0u^0 \in \Omega_{G_{n-1}}$ so that

$$\begin{aligned} d_n(u^{-}tu) &= D_n(u^{-}tu)d_{n-1}(u^{-}tu) \\ &= D_n(t)d_{n-1}(u^{-0}t^0u^0) \end{aligned}$$

is non zero, (considering $K_+[G_{n-1}] \subset K_+[G_n]$).

To show these are the only elements of G_n satisfying this we must use (1.1b) which states that G_n is the disjoint union

$$\bigcup_{w \in W_{G_n}} U^{-}_{G_n}.w.T_{G_n}.U_{G_n}$$

where W_{G_n} is the group of permutation matrices.

Notice that Ω_{G_n} is the term with $w = 1 \in W_{G_n}$.

Case (i)

If $1 \neq w \in W_{G_n}$ has a 1 in the $(n,n)^{th}$ place, then $1 \neq w^0 \in W_{G_{n-1}}$ and therefore by induction $d_{n-1}(u^{-0}w^0t^0u^0) = 0$ so that $d_{n-1}(u^{-}wtu) = 0$ by (1.2b) and hence $d_n(u^{-}wtu) = 0$.

Case (ii)

If $1 \neq w \in W_{G_n}$ has a zero in the $(n,n)^{th}$ place then w^0 is singular so that $D_{n-1}(w^o) = 0$. It now follows that $(u^{-}wtu)^o = u^{-o}w^o t^o u^o$ is singular so that $D_{n-1}(u^{-}wtu) = 0$ and therefore $d_n(u^{-}wtu) = 0$.

Hence, $\Omega_{G_n} = \{g \in G_n : d_n(g) \neq 0\}$.

1.2d Remark

Before proceeding further we make some general remarks concerning rings of fractions and refer the reader to [A.M,§3] for more details.

Let R be any commutative ring with identity and S be a multiplicatively closed subset containing the identity. We define an equivalence relation $\sim$ on $R \times S$ as follows:

$$(r,s) \sim (r_1,s_1) \text{ iff } (rs_1 - r_1s)s' = 0 \text{ for some } s' \in S .$$

We denote by r/s the equivalence class of (r,s) and give the set $R[S^{-1}]$ of all these classes a commutative ring structure:

$$\frac{r}{s} \cdot \frac{r_1}{s_1} = \frac{rr_1}{ss_1} \quad , \quad \frac{r}{s} + \frac{r_1}{s_1} = \frac{rs_1+r_1s}{ss_1} \quad .$$

There is a ring homomorphism $R \to R[S^{-1}]$ given by $r \to r/1_R$. In general this is not injective, but if every element of S is not a zero divisor then $r/1_R = r_1/1_R$ iff $(r-r_1)s = 0$ for some $s \in S$, ie. $r-r_1 = 0$. Thus R can be identified with a subring of $R[S^{-1}]$ in this case and we write r for $r/1_R$.

If R is a K-algebra then $R[S^{-1}]$ inherits from R a K-algebra structure in the obvious manner. If S is generated by a single element $s \in S$ we write $R[s^{-1}]$ for $R[S^{-1}]$.

1.2e Example

Consider the associative K-algebra $K_+[G_n][d_n^{-1}]$. Since $K_+[G_n]$ is an integral domain (it is a free polynomial ring) it can be considered as a K-subalgebra of $K_+[G_n][d_n^{-1}]$. Also every element of $K_+[G_n][d_n^{-1}]$ is of the form q/d_n^m for some $q \in K_+[G_n]$ and non zero integer m, (though not uniquely).

When no confusion should arise we write Ω for Ω_G and d for d_n.

1.2f

We define $\Phi : K_+[G][d^{-1}] \to K^\Omega$ as follows:

let $q \in K_+[G]$ and $m \in Z_{>0}$, then

$$\Phi(q/d^m) : g \to q(g)/d^m(g) \quad \text{for all } g \in \Omega .$$

We must show Φ is well defined.

Suppose that $q/d^m = q_1/d^{m_1}$ for some $q, q_1 \in K_+[G]$ and $m, m_1 \in Z_{>0}$. Then $qd^{m_1} = q_1 d^m$ iff $q(g)d^{m_1}(g) = q_1(g)d^m(g)$ for all $g \in G$ iff $q(g)d^{m_1}(g) = q_1(g)d^m(g)$ for all $g \in \Omega$ (since, if $g \notin \Omega$ then $d^m(g) = d^{m_1}(g) = 0$) iff $q(g)/d^m(g) = q_1(g)/d^{m_1}(g)$ for all $g \in \Omega$ ie. $\Phi(q/d^m) = \Phi(q_1/d^{m_1})$. Thus, Φ is well defined and we have also shown it is injective. Clearly it is a K-algebra morphism, so henceforth we identify $K_+[G][d^{-1}]$ with its image under Φ in K^Ω.

1.2g

We now consider functions on the product $U_G^- \times T_G \times U_G$. It is a standard procedure to show that the K-algebra $A = K[U_G^-] \otimes K[T_G] \otimes K[U_G]$ can be identified with a K-subalgebra of $K^{U_G^- \times T_G \times U_G}$ via the map which takes $f_1 \otimes f_2 \otimes f_3 \in A$ to the function mapping (u^-,t,u) to $f_1(u^-)f_2(t)f_3(u)$. For a proof that this is a well defined injective K-algebra morphism see for example [H, §2.4]. We identify A with its image in $K^{U_G^- \times T_G \times U_G}$.

Now, write $\delta(\delta_n)$ for the element $(1 \otimes d_T \otimes 1) \in A$, where $d_T \in K[T_G]$ is the restriction of $d \in K_+[G]$ to T_G (in fact $d_T = x_{(n,n-1,\ldots,1),K}$). Then, since we consider A as a K-subalgebra of $K^{U_G^- \times T_G \times U_G}$, it is easy to show that $A[\delta^{-1}]$ can be identified with a K-subalgebra of $K^{U_G^- \times T_G \times U_G}$ via the map which takes $q/\delta^m \in A[\delta^{-1}]$ to the function mapping (u^-,t,u) to $q(u^-,t,u)/\delta^m(u^-,t,u)$. Henceforth we identify $A[\delta^{-1}]$ with its image in $K^{U_G^- \times T_G \times U_G}$. Also, since A is an integral domain (it is isomorphic to a free polynomial ring) we can identify it with a subalgebra of $A[\delta^{-1}]$.

1.2h

Define a map $\phi_n : U_{G_n}^- \times T_{G_n} \times U_{G_n} \to \Omega_{G_n}$ by $\phi_n(u^-,t,u) = u^-tu$. Clearly this is surjective. Suppose that $\phi_n(u^-,t,u) = \phi_n(u_1^-,t_1,u_1)$, then $u^-tu = u_1^-t_1u_1$. Hence $(u_1^-)^{-1}u^-t = t_1u_1u^{-1}$ and since the L.H.S. is a lower triangular matrix and the R.H.S. is an upper triangular matrix we see that $t = t_1$, $u^- = u_1^-$ and $u = u_1$. Thus ϕ_n is also injective.

We can now prove the following famous theorem:

1.2i Theorem (Chevalley)

The map $\phi_n : U^-_{G_n} \times T_{G_n} \times U_{G_n} \to \Omega_{G_n}$ induces an isomorphism of K-algebras

$$\phi_n^* : K_+[G_n][d_n^{-1}] \to (K[U^-_{G_n}] \otimes K[T_{G_n}] \otimes K[U_{G_n}])[\delta_n^{-1}]$$

such that $\phi_n^*(1/d_n) = 1/\delta_n$.

Proof

Given $f \in K_+[G_n][d_n^{-1}]$ we define $\phi_n^*(f) \in K^{U^-_{G_n} \times T_{G_n} \times U_{G_n}}$ to be the composition $f \circ \phi_n$, then $\phi_n^*(f)(u^-,t,u) = f(u^- tu)$.

We must first show that the image of ϕ_n^* lies in $A_n[\delta_n^{-1}] = (K[U^-_{G_n}] \otimes K[T_{G_n}] \otimes K[U_{G_n}])[\delta_n^{-1}]$.

Now, since ϕ_n^* is clearly a K-algebra map we need only consider $f = c_{\mu\nu}$ $(\mu,\nu \in \underline{n})$ and $f = 1/d_n$.

(i) If $f = c_{\mu\nu}$ $(\mu,\nu \in \underline{n})$ then $\phi_n^*(c_{\mu\nu})(u^-,t,u) = c_{\mu\nu}(u^- tu)$

$$= \sum_{\rho \in \underline{n}} c^{U^-}_{\mu\rho}(u^-) c^T_{\rho\rho}(t) c^U_{\rho\nu}(u) \text{ . Hence,}$$

$$\phi_n^*(c_{\mu\nu}) = \sum_{\rho \in \underline{n}} c^{U^-}_{\mu\rho} \otimes c^T_{\rho\rho} \otimes c^U_{\rho\nu} \in A_n[\delta_n^{-1}] \text{ .}$$

(ii) If $f = 1/d_n$ then $\phi_n^*(1/d_n)(u^-,t,u) = 1/d_n(u^- tu) = 1/d_n(t)$.

Hence $\phi_n^*(1/d_n) = 1/1 \otimes d_n^T \otimes 1 = 1/\delta_n$.

It follows that $\phi_n^*(K_+[G_n][d_n^{-1}]) \subset A_n[\delta_n^{-1}]$.

Now ϕ_n is surjective, so that ϕ_n^* is injective since if $\phi_n^*(f) = 0$ then $\phi_n^*(f)(u^-,t,u) = f(u^-tu) = 0$ for all $u^-tu \in \Omega_{G_n}$ ie. $f = 0$.

To prove the theorem it remains to show that ϕ_n^* is surjective. We proceed by induction on n. If $n = 1$ then $G_1 = T_{G_1}$ and $K[G_1] = [d_1^{-1}]$ is certainly isomorphic via ϕ_1^* to $(1 \otimes K[T_{G_1}] \otimes 1)[\delta_1^{-1}]$. Suppose that the theorem is true for $n-1$. Now, as remarked above, we can consider $K_+[G_{n-1}]$ as a K-subalgebra of $K_+[G_n]$. We then have an inclusion $i : K_+[G_{n-1}][d_{n-1}^{-1}] \to K_+[G_n][d_n^{-1}]$ such that $i(c_{\mu\nu}) = c_{\mu\nu}$ $(\mu,\nu \in \underline{n-1})$ and $i(1/d_{n-1}) = D_n/d_n$. Clearly we can also consider $K[U^-_{G_{n-1}}]$, $K[T_{G_{n-1}}]$ and $K[U_{G_{n-1}}]$ as K-subalgebras of $K[U^-_{G_n}]$, $K[T_{G_n}]$ and $K[U_{G_n}]$ respectively. Thus we obtain an inclusion $j : A_{n-1}[\delta_{n-1}^{-1}] \to A_n[\delta_n^{-1}]$ such that $j(c^{U^-}_{\mu\nu} \otimes 1 \otimes 1) = c^{U^-}_{\mu\nu} \otimes 1 \otimes 1$, $j(1 \otimes c^T_{\mu\nu} \otimes 1) = 1 \otimes c^T_{\mu\nu} \otimes 1$, $j(1 \otimes 1 \otimes c^U_{\mu\nu}) = 1 \otimes 1 \otimes c^U_{\mu\nu}$ $(\mu,\nu \in \underline{n-1})$ and $j(1/\delta_{n-1}) = (1 \otimes x_{(1,1,\dots,1)} \otimes 1)/\delta_n$. Now, it is easy to see that $\phi_n^* i = j\phi_{n-1}^* : K_+[G_{n-1}][d_{n-1}^{-1}] \to A_n[\delta_n^{-1}]$. By induction ϕ_{n-1}^* is an isomorphism and it then follows that $c^{U^-}_{\mu\nu} \otimes 1 \otimes 1$ $(\mu,\nu \in \underline{n-1})$, $1 \otimes x_\lambda \otimes 1$ $(\lambda \in \Lambda(G_n) : \lambda_n = 0)$ and $1 \otimes 1 \otimes c^U_{\mu\nu}$ $(\mu,\nu \in \underline{n-1})$ are all in the image of ϕ_n^*. Further, since $\phi_n^* i(1/d_{n-1}) = (1 \otimes x_{(1,1,\dots,1)} \otimes 1)/\delta_n$ then $1 \otimes x_{(0,0,\dots 0,1)} \otimes 1 = (1 \otimes \delta_{(1,1,\dots,1)} \otimes 1)^2\, \delta_{n-2}/\delta_n$ is in the image of ϕ_n^*. Hence every $1 \otimes x_\lambda \otimes 1$ $(\lambda \in \Lambda(G_n))$ is in this image.

It remains to show that $c^{U^-}_{n,\nu} \otimes 1 \otimes 1$ $(\nu \in \underline{n})$ and $1 \otimes 1 \otimes c^U_{\mu,n}$ $(\mu \in \underline{n})$ are in the image of ϕ_n^*. Now

$$\phi_n^*(c_{1n}) = \sum_{\rho \in \underline{n}} c^{U^-}_{1\rho} \otimes c^T_{\rho\rho} \otimes c^U_{\rho n}$$

but $c^{U^-}_{1\rho} = 0$ when $\rho > 1$ and $c^{U^-}_{11} = 1$. Thus, we can rewrite $\phi^*_n(c_{1n})$ as $1 \otimes c^T_{11} \otimes c^U_{1n}$ and since there exists a $\lambda \in \Lambda(G_n)$ such that

$$(1 \otimes x_\lambda \otimes 1/\delta_n)(u^-,t,u) = c^T_{11}(t)^{-1}$$

we see that $1 \otimes 1 \otimes c^U_{1n}$ is in the image of ϕ^*_n .

Similarly

$$\phi^*_n(c_{2n}) = c^{U^-}_{21} \otimes c^T_{11} \otimes c^U_{1n} + 1 \otimes c^T_{22} \otimes c^U_{2n}$$

and since $c^{U^-}_{21} \otimes c^T_{11} \otimes c^U_{1n}$ is now shown to be in the image of ϕ^*_n and there exist a $\lambda \in \Lambda(G_n)$ such that

$$(1 \otimes x_\lambda \otimes 1/\delta_n)(u^-,t,u) = c^T_{22}(t)^{-1}$$

we see that $1 \otimes 1 \otimes c^U_{2n}$ is also in the image.

Proceeding in this way we can show that $1 \otimes 1 \otimes c^U_{\mu n}$ is in the image of ϕ^*_n for all $\mu \in \underline{n}$ and by a similar method $c^{U^-}_{n\nu} \otimes 1 \otimes 1$ $(\nu \in \underline{n})$ are also in the image.

This completes the proof.

<u>1.2j</u> <u>Corollary</u> to the proof of (1.2i)

Let $K = Q$ and define $Z_+[G_Q]$ to be the subring of $Q_+[G_Q]$ Z-spanned by monomials in the coordinate functions $c_{\mu\nu}$ $(\mu,\nu \in \underline{n})$ and $Z[U^-_{G(Q)}]$, $Z[T_{G(Q)}]$ and $Z[U_{G(Q)}]$ to be the images of $Z_+[G_Q]$ under the respective restriction maps. Then

$$\phi^*(Z_+[G_Q]) = (Z[U^-_{G(Q)}] \otimes Z[T_{G(Q)}] \otimes Z[U_{G(Q)}])[\delta^{-1}] .$$

Proof

Let $K = Q$ in (1.2i) and observe that the proof works over Z .

1.2k Remark

The proof of (1.2i) differs from Chevalley's original proof in so far as our approach uses precise knowledge of the two K-algebras involved rather than the actions of generating elements of the group on certain modules. Of course it is not surprising that we can do this for the relatively straightforward group G , but we shall also use this approach on the odd orthogonal group in the next section. These proofs underline our approach, that is we are defining our groups as the zeros of a set of polynomials rather than groups generated by certain matrices.

1.3 The Big Cell and Chevalley's theorem for $O_{2\ell+1}$

Henceforth unless stated otherwise we shall assume that the characteristic of K is not 2 . The reason behind this restriction is explained in Remark (1.3h).

We define the following subgroups of the odd orthogonal group $\Gamma = \Gamma_n(K)$ $(n=2\ell+1,\ \ell>0)$:

$$U^-_\Gamma := U^-_G \cap \Gamma ,\quad T_\Gamma := T_G \cap \Gamma ,\quad U_\Gamma := U_G \cap \Gamma$$

and the subset $\Omega_\Gamma := \Omega_G \cap \Gamma$, (the Big Cell).

1.3a Lemma

The subset Ω_Γ is equal to

(i) $\{g\in\Gamma | d_\Gamma(g) \neq 0\}$ where $d_\Gamma \in K[\Gamma]$ is the restriction of $d \in K_+[G]$ to Γ ,

(ii) $U_\Gamma^- T_\Gamma U_\Gamma$.

Proof

Since $\Omega_G = \{g \in G | d(g) \neq 0\}$ (1.2c) then (i) is immediate. To prove (ii) we introduce an automorphism $\sigma: G \to G$ defined by $\sigma(g) = J(g^t)^{-1}J$. It is not hard to show that σ preserves U_G^- , T_G and U_G and that for $g\in G$, $\sigma(g) = g$ if and only if $g\in\Gamma$ (0.4b). Thus, Γ is the set of fixed points of σ . Now, $\Omega_G = U_G^- T_G U_G$ so suppose that $g = u^- tu \in \Omega_G \cap \Gamma$ then $\sigma(g) = \sigma(u^-)\sigma(t)\sigma(u) = g$. But we have shown that every element of Ω_G has a unique expression as a product of elements from U_G^- , T_G and U_G (1.2h). Hence $\sigma(u^-) = u^-$, $\sigma(t) = t$ and $\sigma(u) = u$ so that $u^- \in U_\Gamma^-$, $t \in T_\Gamma$ and $u \in U_\Gamma$. Thus $\Omega_\Gamma \subset U_\Gamma^- T_\Gamma U_\Gamma$ and the reverse inclusion is obvious.

We investigate the elements of U_Γ :

Let Φ be the $(n\times n)$ matrix:

$$\begin{vmatrix} A & D & B \\ 0 & 1 & E \\ 0 & 0 & C \end{vmatrix}$$

A,C are $\ell\times\ell$ upper triangular matrices

B is a $\ell\times\ell$ matrix

D is a $\ell\times 1$ matrix

E is a $1\times\ell$ matrix.

Then $\Phi \in U_\Gamma$ if and only if $\Phi^t J_n \Phi = J_n$ ie.

$$\begin{vmatrix} A^t & 0 & 0 \\ D^t & 1 & 0 \\ B^t & E^t & C^t \end{vmatrix} \begin{vmatrix} 0 & 0 & J_\ell \\ 0 & 1 & 0 \\ J_\ell & 0 & 0 \end{vmatrix} \begin{vmatrix} A & D & B \\ 0 & 1 & E \\ 0 & 0 & C \end{vmatrix} = \begin{vmatrix} 0 & 0 & J_\ell \\ 0 & 1 & 0 \\ J_\ell & 0 & 0 \end{vmatrix}$$

Expanding this we obtain the following relations:

1.3b

(R1) $A^t J_\ell C = J_\ell$

(R2) $E + D^t J_\ell C = 0$

(R3) $B^t J_\ell C + E^t E + C^t J_\ell B = 0$

(R4) $C^t J_\ell A = J_\ell$

(R5) $C^t J_\ell D + E^t = 0$.

It is clear that (R1) is equivalent to (R4) and (R2) to (R5). Thus (R1), (R2) and (R3) are necessary and sufficient conditions for Φ to belong to Γ . Now, if we choose A arbitrarily (upper triangular unipotent) we then determine C by (R1). Further, if we choose D arbitrarily we determine E by (R2). We must now check that (R3) does not interfere with our freedom to choose A and D arbitrarily and see to what extent we are free to choose B :

Let $F = (f_{\mu\nu})_{\mu,\nu\in\underline{\ell}} = B^t J_\ell C$ and $H = (h_{\mu\nu})_{\mu,\nu\in\underline{\ell}} = E^t E$. Then (R3) can be rewritten as:

$(R3)^1$ $F + F^t = -H$

giving $f_{\mu\nu} + f_{\nu\mu} = -h_{\mu\nu} \quad (\mu,\nu\in\underline{\ell})$.

Hence, each $f_{\mu\mu}$ $(\mu\in\underline{\ell})$ is determined by A and D and if we choose $f_{\mu\nu}$ $(\mu<\nu)$ arbitrarily, A and D determine each $f_{\mu,\nu}$ $(\mu\geq\nu)$, but there is no relation between A and D .

Now, since $B = (b_{\mu\nu})_{\mu,\nu\in\underline{\ell}} = J_\ell C^{t^{-1}} F^t$ we can choose the $b_{\mu\nu}$ $(\mu+\nu<\ell)$ arbitrarily, the rest being determined by A and D via C and F .

We can now prove:

<u>1.3c</u> <u>Proposition</u>

The following set of ℓ^2 functions on U_Γ freely generate the K-algebra $K[U_\Gamma]$:

<u>1.3d</u>

$$\{c_{\mu\nu}^{U_\Gamma} : (\mu,\nu) \in R_\ell\}$$

where $c_{\mu\nu}^{U_\Gamma} \in K[U_\Gamma]$ is the restriction of $c_{\mu\nu} \in K_+[G]$ to U_Γ and $R_\ell = \{(\mu,\nu) \in \underline{n}\times\underline{n} : 1 \leq \mu \leq \ell \ , \ \mu+1 \leq \nu \leq 2\ell+1-\mu\}$.

We can represent these elements on a schematic matrix $(\ell = 3)$

$$\left|\begin{array}{ccc|c|ccc}
1 & \ast & \ast & \ast & \ast & \ast & \\
 & 1 & \ast & \ast & \ast & & \\
 & & 1 & \ast & & & \\
\hline
 & & & 1 & & & \\
\hline
 & & & & 1 & & \\
 & & & & & 1 & \\
 & & & & & & 1
\end{array}\right|$$

$\ast$ = element of (1.3d)

Proof

The functions (1.3d) all lie in $K[U_\Gamma]$, since it is certainly generated by the set of functions

1.3e

$$\{c_{\mu\nu}^{U_\Gamma} : 1 \leq \mu < \nu \leq n\} \quad .$$

The calculations above on Φ show that if arbitrary values in K are assigned to each of the functions in (1.3d) then there is a uniquely defined element of U_Γ on which they take these values. This shows that the functions (1.3d) are algebraically independent and that every function in (1.3e) is a polynomial over K in them. This completes the proof.

We can also prove in a similar way:

1.3f Proposition

The following set of ℓ^2 functions on U_Γ^- freely generate the K-algebra $K[U_\Gamma^-]$:

1.3g

$$\{c_{\mu\nu}^{U_\Gamma^-} : (\mu,\nu) \in R_\ell^-\}$$

where $c_{\mu\nu}^{U_\Gamma^-} \in K[U_\Gamma^-]$ is the restriction of $c_{\mu\nu} \in K_+[G]$ to U_Γ^- and $R_\ell^- = \{(\mu,\nu) \in \underline{n}\times\underline{n} : 1 \leq \nu \leq \ell \ , \ \nu+1 \leq \mu \leq 2\ell+1-\nu\}$.

1.3h Remark

Recall that above we had the relation $(R3)^1 : F + F^t = -H$ which yields $2f_{\mu\mu} = -h_{\mu\mu}$ $(\mu\epsilon\underline{n})$. If the characteristic of K is 2 this implies that all of the entries in E are zero. Then (R2) implies that D is determined by A . On the other hand the entries $\{b_{\mu\nu} : \mu+\nu \leq \ell\}$ can be chosen arbitrarily (ℓ more than before). All this means that $K[U_\Gamma]$ is no longer freely generated by the ℓ^2 functions (1.3d) but by the ℓ^2 functions

$$\{c_{\mu\nu}^{U_\Gamma} : 1\leq\mu\leq\ell,\ \mu+1\leq\nu\leq\ell,\ \ \ell+2\leq\nu\leq 2\ell+2-\mu\} .$$

It is to avoid this complication that we assume the characteristic of K is not 2.

1.3i Remark

If $K = Q$, observe that each $c_{\mu\nu}^{U_-}(\mu,\nu\epsilon\underline{n})$ lies in $Z^*[c_{\mu\nu}^{U_\Gamma}:(\mu,\nu)\epsilon R_\ell]$, where Z^* is the subring of Q generated by Z and $\frac{1}{2}$. The need to adjoin $\frac{1}{2}$ to Z arises from the relation $2f_{\mu\mu} = -h_{\mu\mu}$ $(\mu\epsilon\underline{n})$, (cf. (1.3h)).

We now turn our attention to T_Γ . It is readily seen that a diagonal matrix $\mathrm{diag}(t_1,t_2,\ldots,t_n)$ lies in Γ if and only if $t_\mu t_{n+1-\mu} = 1$ $(\mu\epsilon\underline{n})$. Thus $t_1,t_2,\ldots,t_\ell$ can be chosen arbitrarily non zero and $t_{\ell+1}^2 = 1$. It follows that $K[T_\Gamma]$ has a basis of monomials

1.3j
$$(c_{1,1}^{T_\Gamma})^{\alpha_1}(c_{2,2}^{T_\Gamma})^{\alpha_2}\ldots(c_{\ell,\ell}^{T_\Gamma})^{\alpha_\ell}\,(c_{\ell+1,\ell+1}^{T_\Gamma})^{\alpha_{\ell+1}}$$

where $\alpha_1,\alpha_2,\ldots,\alpha_\ell \epsilon Z$, $\alpha_{\ell+1}\epsilon\{0,1\}$ and $c_{\mu\mu}^{T_\Gamma} \epsilon K[T_\Gamma]$ denotes the restriction of $c_{\mu\mu} \epsilon K_+[G]$ to T_Γ .

Denote by $\Lambda(\Gamma)$ the set of $\ell+1$-tuples $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_{\ell+1})$ such that $\alpha_1,\alpha_2,\ldots,\alpha_\ell \in Z$ and $\alpha_{\ell+1} \in \{0,1\}$. As with T_G we can define the character $\chi^\Gamma_{\alpha,K}$ of Γ given by (1.3j). Thus

1.3k

$$\chi^\Gamma_{\alpha,K}(\mathrm{diag}(t_1,t_2,\ldots,t_\ell,\varepsilon,t_\ell^{-1},\ldots,t_1^{-1}) = t_1^{\alpha_1}t_2^{\alpha_2}\ldots t_\ell^{\alpha_\ell}\varepsilon^{\alpha_{\ell+1}}$$

where $\varepsilon = \pm 1$.

We call elements of $\Lambda(\Gamma)$ 'weights' and $\chi^\Gamma_{\alpha,K}$ $(\alpha\in\Lambda(\Gamma))$ a 'character of weight α'. Clearly $K[T_\Gamma]$ has a basis $\{\chi^\Gamma_{\alpha,K} : \alpha \in\Lambda(\Gamma)\}$.

Following (1.2f) and (1.2g) we can identify the K-algebras $K[\Gamma][d_\Gamma^{-1}]$ and $A_\Gamma = K[U_\Gamma^-] \otimes K[T_\Gamma] \otimes K[U_\Gamma]$ with K-subalgebras of K^{Ω_Γ} and $K^{U_\Gamma^-\times T_\Gamma\times U_\Gamma}$ respectively. Notice however that $K[\Gamma]$ is no longer an integral domain. Also, the counterpart of δ, $\delta_\Gamma := (1 \otimes d_{T_\Gamma} \otimes 1) \in A_\Gamma$ is invertible since the restriction d_{T_Γ} of $d \in K_+[G]$ to T_Γ is the character $\chi^\Gamma_{(n-1,n-3,\ldots,0)}$ which has inverse $\chi^\Gamma_{(1-n,3-n,\ldots,0)}$ in A_Γ . Thus $A_\Gamma[\delta^{-1}] = A_\Gamma$.

We also have the analogue of (1.2h):

1.3ℓ

The map $\phi_\Gamma : U_\Gamma^- \times T_\Gamma \times U_\Gamma \to \Omega_\Gamma$ defined by $\phi_\Gamma(u^-,t,u) = u^-tu$ is a bijection.

We can now prove Chevalley's theorem for Γ .

1.3m **Theorem** (Chevalley)

The map $\phi_\Gamma : U_\Gamma^- \times T_\Gamma \times U_\Gamma \to \Omega_\Gamma$ induces an isomorphism of K-algebras:

$$\phi_\Gamma^* : K[\Gamma][d_\Gamma^{-1}] \to K[U_\Gamma^-] \otimes K[T_\Gamma] \otimes K[U_\Gamma] \ .$$

Proof

Given $f \in K[\Gamma][d_\Gamma^{-1}]$ we define $\phi_\Gamma^*(f) \in K^{U_\Gamma^- \times T_\Gamma \times U_\Gamma}$ to be the composition $f \circ \phi_\Gamma$, then $\phi_\Gamma^*(f)(u^-,t,u) = f(u^-tu)$.

As in the proof of (1.2i) ϕ_Γ^* is an injective K-algebra morphism into A_Γ . We have only to show that ϕ_Γ^* is surjective.

Take any element $a \in A_\Gamma$, then there exists an element $\hat{a} \in (K[U_G^-] \otimes K[T_G] \otimes K[U_G])[\delta^{-1}]$ such that $\hat{a}$ restricted to $U_\Gamma^- \times T_\Gamma \times U_\Gamma$ is precisely a , (of course $\hat{a}$ is not uniquely determined by a). By (1.2i) there exists $\hat{q} \in K_+[G]$ and $m \in Z_{\geq 0}$ such that $\phi^*(\hat{q}/d^m) = \hat{a}$ ie.

$$\frac{\hat{q}(u^-tu)}{d^m(u^-tu)} = \hat{a}(u^-,t,u)$$

for all $(u^-,t,u) \in U_G^- \times T_G \times U_G$.

This equation holds in particular for all $(u^-,t,u) \in U_\Gamma^- \times T_\Gamma \times U_\Gamma$ and it follows that

$$\phi_\Gamma^*(q/d_\Gamma^m) = a$$

where $q \in K[\Gamma]$ is the restriction of $\hat{q} \in K_+[G]$ to Γ .

This completes the proof.

<u>1.3n</u> <u>Corollary</u> to the proof of (1.3m) and (1.2j).

Let $K = Q$ and define $Z[\Gamma_Q]$, $Z[U^-_{\Gamma(Q)}]$, $Z[T_{\Gamma(Q)}]$ and $Z[U_{\Gamma(Q)}]$ to be the images of $Z_+[G_Q]$ under the respective restriction maps.

Then

$$\phi^*_\Gamma(Z[\Gamma_Q][d^{-1}_{\Gamma_Q}]) = Z[U^-_{\Gamma(Q)}] \otimes Z[T_{\Gamma(Q)}] \otimes Z[U_{\Gamma(Q)}] \ .$$

<u>Proof</u>

Let $K = Q$ in (1.3m) and observe the proof works over Z .

§2. The kernel of $\Psi_K : K_+[G] \to K[\Gamma]$

2.1 Preliminaries

In this chapter we shall find a generating set for the kernel of the restriction map $\Psi_K : K_+[G] \to K[\Gamma]$ as an ideal of $K_+[G]$. However, our first move shall be to compute the kernel of the restriction map

2.1a

$$\tilde{\Psi}_K^{U^- \times T \times U} : K[U_G^-] \otimes K[T_G] \otimes K[U_G] \to K[U_\Gamma^-] \otimes K[T_\Gamma] \otimes K[U_\Gamma]$$

where $\tilde{\Psi}_K^{U^- \times T \times U} = \Psi_K^{U^-} \otimes \Psi_K^T \otimes \Psi_K^U$ and

$\Psi_K^{U^-}$ is restriction $K[U_G^-] \to K[U_\Gamma^-]$,

Ψ_K^T is restriction $K[T_G] \to K[T_\Gamma]$ and

Ψ_K^U is restriction $K[U_G] \to K[U_\Gamma]$.

We have:

2.1b

$$\text{Ker } \tilde{\Psi}_K^{U^- \times T \times U} = \ker \Psi_K^{U^-} \otimes K[T_G] \otimes K[U_G] + K[U_G^-] \otimes \ker \Psi_K^T \otimes K[U_G] + K[U_G^-] \otimes K[T_G] \otimes \ker \Psi_K^U .$$

Hence it is enough to consider the kernels of $\Psi_K^{U^-}$, Ψ_K^T and Ψ_K^U separately, this shall be done in (2.2) and (2.3). We shall then use the isomorphisms given by Chevalley's theorem (1.2i, 1.3m) to achieve our aim. Before we commence, a definition:

2.1c Definition

For any subgroup G of G and $\alpha,\beta \in \underline{n}$ we define the following elements of $K[G]$:

$$F^G_{\alpha\beta} = \sum_{\mu\in\underline{n}} c^G_{\mu\alpha}\, c^G_{\overline{\mu}\beta} - \delta_{\alpha,\overline{\beta}}\, 1_{K[G]} \ ,$$

$$H^G_{\alpha\beta} = \sum_{\mu\in\underline{n}} c^G_{\alpha\mu}\, c^G_{\beta\overline{\mu}} - \delta_{\alpha,\overline{\beta}}\, 1_{K[G]} \ ,$$

where for $\nu\in\underline{n}$, $\overline{\nu} := n+1-\nu$ and $\delta_{\alpha,\overline{\beta}} = \begin{cases} 1 & \alpha=\overline{\beta} \\ 0 & \alpha\neq\overline{\beta} \end{cases}$.

Notice that if $C^G = (c^G_{\mu\nu})_{\mu,\nu\in n}$, an $(n\times n)$ matrix, then

2.1d
$$(C^G)^t J_n C^G - J_n = (F^G_{\alpha\beta})_{\alpha,\beta\in n} \quad \text{and}$$

2.1e
$$C^G J_n (C^G)^t - J_n = (H^G_{\alpha\beta})_{\alpha,\beta\in \underline{n}} \ .$$

2.2 The kernel of $\Psi_K^U : K[U_G] \to K[U_\Gamma]$

2.2a Proposition

The kernel of the restriction map

$$\Psi_K^U : K[U_G] \to K[U_\Gamma]$$

is generated as an ideal of $K[U_G]$ by the set of elements

2.2b

$$\{F^U_{\alpha\beta} : \alpha,\beta\in\underline{n} , \bar{\beta} < \alpha\}$$

and also by the elements

2.2c

$$\{H^U_{\alpha\beta} : \alpha,\beta\in\underline{n} , \bar{\beta} > \alpha\} .$$

Proof

Consider the matrix $C^U = (c^U_{\mu\nu})_{\mu,\nu\in\underline{n}}$.

By (2.1d) $F^U_{\alpha\beta}$ $(\alpha,\beta\in\underline{n})$ is the $(\alpha,\beta)^{th}$ coefficient of the matrix $(C^U)^t J_n C^U - J_n$. Let F be the ideal of $K[U_G]$ generated by all these elements. Then, since C^U is an upper triangular unipotent matrix, it is not difficult to see that F is also generated by the set of elements (2.2b). (these are the entries of $(C^U)^t J_n C^U - J_n$ below the diagonal which runs from the top right hand to bottom left hand corners). Now, since for any $g \in U_\Gamma$, $g^t J_n g - J_n$ is a zero matrix it is clear that F is contained in the kernel of $\Psi^U_K : K[U_G] \to K[U_\Gamma]$.

Denote by $\bar{c}_{\mu\nu}$ $(\mu,\nu\in\underline{n})$ the image of $c^U_{\mu\nu}$ in $K[U_G]/F$ and let $\bar{C}$ be the matrix $(\bar{c}_{\mu\nu})_{\mu,\nu\in\underline{n}}$. Then $\bar{C}$ is an upper triangular unipotent matrix and $\bar{C}^t J_n \bar{C} - J_n \equiv 0$. It follows that if we submit $\bar{C}$ to the process described in (§1.3) we shall show, as in the proof of (1.3c), that every coefficient $\bar{c}_{\mu\nu}$ can be expressed as a polynomial in the elements

$\{\bar{c}_{\mu\nu} : (\mu,\nu) \in R_\ell\}$, (though these elements do not necessarily freely generate $K[U_G]/F$). It follows immediately that

$$K[U_G] = K[c^U_{\mu\nu} : (\mu,\nu) \in R_\ell] + F \quad \ldots . \qquad (1)$$

But we also have that

$$K[U_G] = K[c^U_{\mu\nu} : (\mu,\nu) \in R_\ell] \oplus \ker \Psi^U_K \quad \ldots . \qquad (2)$$

the sum being direct since $\{\Psi^U_K(c^U_{\mu\nu}) : (\mu,\nu) \in R_\ell\}$ freely generate the K-algebra $\Psi^U_K(K[U_G]) = K[U_\Gamma]$, (1.3c).

Now, let $z \in \ker \Psi^U_K$, then using (1) we have $z = x+y$ with $x \in K[c^U_{\mu\nu} : (\mu,\nu) \in R_\ell]$ and $y \in F$. But $F \subset \ker \Psi^U_K$, thus $x \in \operatorname{Ker} \Psi^U_K \quad K[c^U_{\mu\nu} : (\mu,\nu) \in R_\ell] = \{0\}$ by (2), and therefore $z \in F$. This proves that the set of elements (2.2b) generates $\ker \Psi^U_K$.

To show that the set of elements (2.2c) also generates the kernel we submit the lower triangular unipotent matrix $(C^U)^t$ to a process similar to that for upper triangular unipotent matrices in (§1.3), and thereby show that

$$K[U_G] = K[c^U_{\nu,\mu} : (\mu,\nu) \in R^-_\ell] + H$$

where H is the ideal of $K[U_G]$ generated by the elements (2.2c). Now, it is easy to see that $(\nu,\mu) \in R^-_\ell$ iff $(\mu,\nu) \in R_\ell$. Hence

$$K[U_G] = K[c^U_{\mu\nu} : (\mu,\nu) \in R_\ell] + H$$

and since for $g \in U_G$, $g^t J_n g = J_n$ iff $g J_n g^t = J_n$ then $H \subset \ker \Psi^U_K$

and the rest of the proof follows as before.

In a similar way we can prove:

2.2d Proposition

The kernel of the restriction map

$$\Psi_K^{U^-} : K[U_G^-] \to K[U_\Gamma^-]$$

is generated as an ideal of $K[U_G^-]$ by the set of elements

2.2e

$$\{F_{\alpha\beta}^{U^-} : \alpha,\beta\in\underline{n} \ , \ \bar{\beta} > \alpha\}$$

and also by the set of elements

2.2f

$$\{H_{\alpha\beta}^{U^-} : \alpha,\beta\in\underline{n} \ , \ \bar{\beta} < \alpha\}$$

2.3 The kernel of $\Psi_K^T : K[T_G] \to K[T_\Gamma]$

We have shown in (1.1c) that $K[T_G]$ is freely generated as a K-algebra by the elements $\{c_{\mu\mu}^T : \mu\in\underline{n}\}$, and in (1.3j) that $K[T_\Gamma]$ is the K-algebra

2.3a

$$K(X_1, X_2, \ldots, X_{\ell+1} : X_{\ell+1}^2 = 1)$$

where $X_\mu := c_{\mu\mu}^{T_\Gamma} = \Psi_K^T(c_{\mu\mu}^T)$ and $X_\mu^{-1} := c_{\bar{\mu}\bar{\mu}}^{T_\Gamma} = \Psi_K^T(c_{\bar{\mu}\bar{\mu}}^T)$ $(\mu\in\underline{\ell+1})$. Thus $X_1, X_2, \ldots, X_\ell$ are indeterminates.

It follows that the kernel of Ψ_K^T is therefore generated as an ideal of $K[T_G]$ by the elements

2.3b

$$c_{\mu\mu}^T \, c_{\bar{\mu}\bar{\mu}}^T - 1_{K[T_G]} \qquad (\mu\epsilon\underline{n})$$

But $H_{\mu\bar{\mu}}^T = F_{\mu\bar{\mu}}^T = c_{\mu\mu}^T \, c_{\bar{\mu}\bar{\mu}}^T - 1_{K[T_G]} \qquad (\mu\epsilon\underline{n})$.

Hence, we have

2.3c Proposition

The kernel of the restriction map

$$\Psi_K^T : K[T_G] \to K[T_\Gamma]$$

is generated as an ideal of $K[T_G]$ by the set of elements

$$\{H_{\alpha\bar{\alpha}}^T = F_{\alpha\bar{\alpha}}^T : \alpha\epsilon\underline{n} , \quad \bar{\alpha} \geq \alpha\} \ .$$

Now, by (2.1b) propositions (2.2a,d) and (2.3c) give generating elements of the kernel of $\Psi_K^{U^- \times T \times U}$ as an ideal of $K[U_G^-] \otimes K[T_G] \otimes K[U_G]$.

2.4 The kernel of $\Psi_K^\Omega : K_1[G][d^{-1}] \to K[\Gamma][d_T^{-1}]$

We prove a general lemma concerning rings of fractions.

2.4a Lemma

Let R and R_1 be commutative rings with identity and $f:R \to R_1$ be an epimorphism of rings. Let S be a multiplicatively closed subset of R containing the identity then

$$0 \to (\ker f)[S^{-1}] \to R[S^{-1}] \xrightarrow{\bar{f}} R_1[f(S)^{-1}] \to 0$$

is exact, where for $r \in R$, $s \in S$

$$\bar{f}(r/s) = f(r)/f(s) \in R_1[f(S)^{-1}] \quad .$$

Proof

We must first check that $\bar{f} : R[S^{-1}] \to R[f(S)^{-1}]$ is well defined. Suppose that $r/s = r'/s'$ in $R[S^{-1}]$, then $(rs'-r's)s'' = 0$ in R for some $s'' \in S$. Thus $(f(r)f(s')-f(r')f(s))f(s'') = 0$ in R_1 so that $f(r)/f(s) = f(r')/f(s')$ in $R_1[f(S)^{-1}]$ giving $\bar{f}(r/s) = \bar{f}(r'/s')$, which shows that $\bar{f}$ is well defined as required.

Now, since f is an epimorphism it follows that $\bar{f}$ is also. It therefore remains to show that $\ker \bar{f} = (\ker f)[S^{-1}] = \{r/s : r \in \ker f,\ s \in S\}$. Suppose that $\bar{f}(r/s) = 0$ then $f(r)/f(s)$ is zero in $R_1[f(S)^{-1}]$ so that there is some $s' \in S$ with $f(r)f(s') = f(rs') = 0$ in R_1 . Then $rs' \in \ker f$ and $r/s = rs'/ss' \in (\ker f)[S^{-1}]$ as required.

We can now construct the following commutative diagram:

2.4b

$$\begin{tikzcd}
(\ker \hat{\Psi}_K^{U^- \times T \times U})[\delta^{-1}] \arrow[r] & A_K[\delta^{-1}] \arrow[r, two heads, "\hat{\Psi}_K^{U^- \times T \times U}"] & A_{\Gamma(K)}[\delta_\Gamma^{-1}] \\
J_K[d^{-1}] \arrow[r] \arrow[u, "\wr"] & K_+[G][d^{-1}] \arrow[r, two heads, "\hat{\Psi}_K^{\Omega}"] \arrow[u, "\phi_K^*", "\wr"'] & K[\Gamma][d_\Gamma^{-1}] \arrow[u, "\phi_{\Gamma(K)}^*", "\wr"'] \\
J_K \arrow[r] \arrow[u] & K_+[G] \arrow[r, two heads, "\Psi_K"] \arrow[u] & K[\Gamma] \arrow[u]
\end{tikzcd}$$

where $A_K = K[U_G^-] \otimes K[T_G] \otimes K[U_G]$

$A_{\Gamma(K)} = K[U_\Gamma^-] \otimes K[T_\Gamma] \otimes K[U_\Gamma]$

J_K = kernel of $\Psi_K : K_+[G] \to K[\Gamma]$

and $d \in K_+[G]$, $\delta \in A_K$ are the elements described in (1.2c,g) and $d_\Gamma \in K[\Gamma]$, $\delta_\Gamma \in A_{\Gamma(K)}$ their counterparts with respect to $\Gamma(K)$. Recall $A_{\Gamma(K)}[\delta_\Gamma^{-1}] = A_{\Gamma(K)}$, (1.3, p.32).

The first and second horizontal rows are exact by (2.4a), $\Psi_K^{U^- \times T \times U}$ and Ψ_K^Ω being the natural maps induced by $\Psi_K^{U^- \times T \times U}$ and Ψ_K respectively. The upper row of vertical isomorphisms arise from Chevalley's theorem (1.2i) and (1.3m) and the bottom row of vertical maps are all monomorphisms since $K_+[G]$ is an integral domain (1.2e) and:

2.4c Lemma

The element $d_\Gamma \in K[\Gamma]$ is not a zero divisor, therefore $K[\Gamma]$ can be identified with a K-subalgebra of $K[\Gamma][d_\Gamma^{-1}]$, (1.2d).

Proof

See appendix A.

2.4d Definition

Let I_K be the ideal of $K_+[G]$ generated, as an ideal, by the set of elements

$$\{F_{\alpha\beta}^G , H_{\alpha\beta}^G : \alpha,\beta \in \underline{n}\} .$$

Define B_K to be the K-algebra $K_+[G]/I_K$, then we have a s.e.s.

2.4e

$$I_K \rightarrow K_+[G] \overset{\theta}{\twoheadrightarrow} B_K$$

where $\theta: K_+[G] \twoheadrightarrow B_K$ is the natural map.

By applying (2.4a) we also have the s.e.s.

2.4f

$$I_K[d^{-1}] \rightarrow K_+[G][d^{-1}] \overset{\bar{\theta}}{\twoheadrightarrow} B_K[d_B^{-1}]$$

where $d_B := \theta(d) \in B_K$.

Clearly $I_K \subset J_K = \ker \Psi_K : K_+[G] \rightarrow K[\Gamma]$. Hence $I_K[d^{-1}] \subset J_K[d^{-1}]$, but in fact we have:

2.4g Proposition

The ideals $I_K[d^{-1}]$ and $J_K[d^{-1}]$ of $K_+[G][d^{-1}]$ are equal.

Proof

It is enough to show equality of the ideals $\phi_K^*(I_K[d^{-1}])$ and $\phi_K^*(J_K[d^{-1}])$ in $A_K[\delta^{-1}]$.

By (2.4b), $\phi_K^*(J_K[d^{-1}]) = (\ker \Psi_K^{U^- \times T \times U})[\delta^{-1}]$ and by (2.1b), (2.2a,d) and (2.3c) the following elements of $A_K \subset A_K[\delta^{-1}]$ generate it as an ideal of $A_K[\delta^{-1}]$:

$$\begin{array}{lll} F_{\alpha\beta}^{U^-} \otimes 1 \otimes 1 & \alpha,\beta \in \underline{n} & \bar{\beta} > \alpha \\ 1 \otimes F_{\alpha\bar{\alpha}}^{T} \otimes 1 & \alpha \in \underline{n} & \bar{\alpha} \geq \alpha \\ 1 \otimes 1 \otimes H_{\alpha\beta}^{U} & \alpha,\beta \in \underline{n} & \bar{\beta} > \alpha \end{array}$$

Let $X = \{(\alpha,\beta) : \alpha,\beta \in \underline{n}\ \ \overline{\beta} \geq \alpha\}$ and totally order it lexicographically so that $(\alpha',\beta') < (\alpha,\beta)$ iff $\alpha' < \alpha$ or $\alpha' = \alpha$ and $\beta' < \beta$.

Now, for $(\alpha,\beta) \in X$ define $I(\alpha,\beta)$ to be the ideal of $A_K[\delta^{-1}]$ generated by the elements

$$\phi_K^*(F^G_{\alpha'\beta'}) \ , \quad \phi_K^*(H^G_{\alpha'\beta'})$$

with $(\alpha',\beta') \in S$ and $(\alpha',\beta') \leq (\alpha,\beta)$.

<u>Claim</u>

For any $(\alpha,\beta) \in X$, $I(\alpha,\beta)$ is the ideal of $A_K[\delta^{-1}]$ generated by the elements

$$F^{U^-}_{\alpha'\beta'} \otimes 1 \otimes 1 \ , \quad 1 \otimes F^T_{\alpha'\beta'} \otimes 1 \ , \quad 1 \otimes 1 \otimes H^U_{\alpha'\beta'}$$

for all $(\alpha',\beta') \in X$, $(\alpha',\beta') \leq (\alpha,\beta)$, (note that $F^T_{\alpha'\beta'} = 0$ unless $\alpha' = \overline{\beta}'$ and $F^{U^-}_{\alpha'\overline{\alpha}'} = 0 = H^U_{\alpha'\overline{\alpha}'}$) .

<u>Proof of Claim</u>

Let $(\alpha,\beta) \in X$, then

$$(i) \qquad \phi_K^*(F^G_{\alpha\beta}) = \Big(\sum_{\mu,h,k\in\underline{n}} c^{U^-}_{\mu h} c^{U^-}_{\overline{\mu} k} \otimes c^T_{hh} c^T_{kk} \otimes c^U_{h\alpha} c^U_{k\beta} \Big) - \delta_{\alpha,\overline{\beta}} \cdot 1_{A_K}$$

Now, since $c^U_{\mu\nu} = 0$ if $\mu > \nu$, the sum can be taken over all $h,k \in \underline{n}$ with $h \leq \alpha$ and $k \leq \beta$.

Consider now the element of $A_K[\delta^{-1}]$

(ii) $$\sum_{\substack{h\le\alpha\\ k\le\beta}} (F^{U^-}_{h,k} \otimes c^T_{hh}c^T_{kk} \otimes c^U_{h\alpha}c^U_{k\beta}) + \delta_{\alpha,\overline{\beta}}(1 \otimes F^T_{\alpha\overline{\alpha}} \otimes 1) \quad .$$

Note that since $\overline{\beta} \ge \alpha$ the conditions $h \le \alpha$, $k \le \beta$ imply $\overline{k} \ge \overline{\beta} \ge \alpha \ge h$. Thus the terms $F^{U^-}_{hk}$ have $(h,k) \in X$ and $(h,k) \le (\alpha,\beta)$ in our ordering. We show this is precisely $\phi^*_K(F^G_{\alpha\beta})$.

Expanding each $F^{U^-}_{hk}$ and $F^T_{\alpha\overline{\alpha}}$ we have:

$$\sum_{\substack{h\le\alpha\\ k\le\beta\\ \mu,h,k\in\underline{n}}} \epsilon\, c^{U^-}_{\mu h}c^{U^-}_{\mu k} \otimes c^T_{hh}c^T_{kk} \otimes c^U_{h\alpha}c^U_{k\beta} - \sum_{\substack{h\le\alpha\\ k\le\beta}} \delta_{h,\overline{k}}(1\otimes c^T_{hh}c^T_{kk}\otimes c^U_{h\alpha}c^U_{k\beta})$$

$$+ \delta_{\alpha\overline{\beta}}(1 \otimes c^T_{\alpha\alpha}c^T_{\overline{\alpha}\overline{\alpha}} \otimes 1 - 1 \otimes 1 \otimes 1) \quad .$$

Now $\delta_{h,\overline{k}} = 0$ unless $h = \overline{k}$, but then $\alpha = h$ and $k = \beta$ since $\overline{k} \ge \overline{\beta} \ge \alpha \ge h$. Thus the second sum is $\delta_{\alpha\overline{\beta}}(1 \otimes c^T_{\alpha\alpha}c^T_{\overline{\alpha}\overline{\alpha}} \otimes 1)$ and this shows that $\phi^*_K(F^G_{\alpha\beta})$ equals (ii).

Similarly, it can be shown that for $(\alpha,\beta) \in X$

(iii) $$\phi^*_K(H^G_{\alpha\beta}) = \sum_{\substack{h\le\alpha\\ k\le\beta}} (c^{U^-}_{\alpha h}c^{U^-}_{\beta k} \otimes c^T_{hh}c^T_{kk} \otimes H^U_{hk}) + \delta_{\alpha,\overline{\beta}}(1 \otimes F^T_{\alpha\overline{\alpha}} \otimes 1) \quad .$$

We can now prove the claim by induction on $(\alpha,\beta) \in X$. Fix $(\alpha,\beta) \in X$ and denote by (α_0,β_0) the element of X next below (α,β) in our ordering. Assume the claim is true for $I(\alpha_0,\beta_0)$ then by the identities (ii) and (iii) we have, modulo $I(\alpha_0,\beta_0)$,

$$\phi^*_K(F^G_{\alpha\beta}) \equiv F^{U^-}_{\alpha\beta} \otimes c^T_{\alpha\alpha}c^T_{\beta\beta} \otimes 1 + \delta_{\alpha\overline{\beta}}(1 \otimes F^T_{\alpha\overline{\alpha}} \otimes 1)$$

and

$$\phi_K^*(H_{\alpha\beta}^G) = 1 \otimes c_{\alpha\alpha}^T c_{\beta\beta}^T \otimes H_{\alpha\beta}^U + \delta_{\alpha\bar{\beta}} (1 \otimes F_{\alpha\bar{\alpha}}^T \otimes 1) \quad .$$

Hence, $I(\alpha,\beta)$ is ideal-generated modulo $I(\alpha_0,\beta_0)$ by

$$F_{\alpha\beta}^{\bar{U}} \otimes 1 \otimes 1 \quad \text{and} \quad 1 \otimes 1 \otimes H_{\alpha\beta}^{\bar{U}} \quad \text{in the case } \bar{\beta} > \alpha$$

(using the fact that $(1 \otimes c_{\alpha\alpha}^T c_{\beta\beta}^T \otimes 1)^{-1} \in A_K[\delta^{-1}]$, (1.3j)), and by $1 \otimes F_{\alpha\bar{\alpha}}^T \otimes 1$ in the case $\alpha = \bar{\beta}$, (using the fact that $F_{\alpha\alpha}^{\bar{U}} = 0 = H_{\alpha\bar{\alpha}}^U$) .

Hence the claim is true for (α,β). It remains only to start the induction. Let $(\alpha,\beta) = (1,1)$, the minimal element of X , then

$$\phi_K^*(F_{11}^G) = F_{11}^{\bar{U}} \otimes c_{11}^T c_{11}^T \otimes 1 \ , \ \phi_K^*(H_{11}^G) = 1 \otimes c_{11}^T c_{11}^T \otimes H_{11}^U$$

and since $(1 \otimes c_{11}^T c_{11}^T \otimes 1)^{-1} \in A_K[\delta^{-1}]$ the proof of the claim is complete.

Now, set $(\alpha,\beta) = (n,1)$, the maximal element of X . Then $I(n,1) \subset \phi_K^*(I_K[d^{-1}])$ by definition and by the claim $I(n,1) = \phi_K^*(J_K[d^{-1}])$ since they are both generated as ideals by the same elements of $A_K \subset A_K[\delta^{-1}]$. Since we already have $I_K[d^{-1}] \subset J_K[d^{-1}]$ it follows that

$$\phi_K^*(I_K[d^{-1}]) = \phi^*(J_K[d^{-1}])$$

proving the theorem.

We have the following important corollary.

2.4h Corollary

There is an isomorphism of K-algebras

$$\Phi : B_K[d_B^{-1}] \to K[\Gamma][d_\Gamma^{-1}]$$

such that for $x \in K_+[G]$, $m \in Z_{\geq 0}$

$$\Phi\ (\theta(x)/d_B^m) = \Psi_K(x)/d_\Gamma^m \quad .$$

Proof

Using (2.4b,f,g) we have the following commutative diagram:

$$\begin{array}{ccccc} I_K[d^{-1}] & \to & K_+[G][d^{-1}] & \xrightarrow{\bar{\theta}} & B_K[d_B^{-1}] \\ \downarrow \text{id} & & \downarrow \text{id} & & \vdots\ \Phi \\ J_K[d^{-1}] & \to & K_+[G][d^{-1}] & \xrightarrow{\Psi_K^{\Omega}} & K[\Gamma][d_\Gamma^{-1}] \end{array} \quad .$$

Hence, there must be an isomorphism Φ between $B_K[d_B^{-1}]$ and $K[\Gamma][d_\Gamma^{-1}]$ given by

$$\Phi(\theta(x)/d_B^m) = \Psi_K(x)/d_\Gamma^m$$

for all $x \in K_+[G]$, $m \in Z_{\geq 0}$.

2.5 The K-algebra B_K

The aim of this section is to show that B_K is a reduced algebra, that is its nilradical, which is the ideal of B_K consisting of all nilpotent elements, is zero. We already have:

2.5a Proposition

The K-algebra $B_K[d_B^{-1}]$ is reduced.

Proof

By (2.4h) and the identification of (§1.3), $B_K[d_B^{-1}]$ is isomorphic to a subalgebra of the algebra of functions K^{Ω_Γ}. Since K^{Ω_Γ} certainly contains no non zero nilpotent elements, the nilradical of $B_K[d_B^{-1}]$ is zero. It is therefore a reduced algebra as required.

We need to show that B_K has a Hopf algebra structure, to this end we prove:

2.5b Lemma

I_K is a coideal of $K_+[G]$ so that B_K inherits a coalgebra structure from $K_+[G]$.

Proof

Recall that in (0.2a) we gave $K_+[G]$ a coalgebra structure with comultiplication

$$\Delta: K_+[G] \to K_+[G] \otimes K_+[G]$$

such that $\Delta(c_{\mu\nu}) = \sum_{\rho\in\underline{n}} c_{\mu\rho} \otimes c_{\rho\nu}$ $(\mu,\nu\in\underline{n})$ and counit $e: K_+[G] \to K$, evaluation at the identity of G.

By definition, I_K is a coideal if

$$\Delta(I_K) \subset K_+[G] \otimes I_K + I_K \otimes K_+[G]$$

and $$e(I_K) = 0 \quad . \quad \text{[Sw, p.18.]}$$

Now, for any $\alpha,\beta \in \underline{n}$ it is clear that

$$F^G_{\alpha\beta}(1_G) = 0 = H^G_{\alpha\beta}(1_G)$$

and it is not hard to show that

$$\Delta(F^G_{\alpha\beta}) = \sum_{\mu,\nu\in\underline{n}} F^G_{\mu\nu} \otimes c_{\mu\alpha}c_{\nu\beta} + (1 \otimes F^G_{\alpha\beta})$$

$$\Delta(H^G_{\alpha\beta}) = \sum_{\mu,\nu\in\underline{n}} c_{\alpha\mu}c_{\beta\nu} \otimes H^G_{\mu\nu} + (H^G_{\alpha\beta} \otimes 1) \quad .$$

Since these elements generate I_K as an ideal of $K_+[G]$ and $\Delta(xy) = \Delta(x)\Delta(y)$, $e(xy) = e(x)e(y)$ for all $x,y \in K_+[G]$ it follows that I_K is a coideal. Then $B_K = K_+[G]/I_K$ can be given a coalgebra structure by defining comultiplication $\Delta_B : B_K \to B_K \otimes B_K$ by

$$\Delta_B(x+I_K) = \Sigma(x_{(1)} + I_K) \otimes (x_{(2)} + I_K)$$

where $\Delta(x) = \Sigma x_{(1)} \otimes x_{(2)}$ $(x \in K_+[G])$ (using the notation of [Sw]), and counit $e_B : B_K \to K$ by

$$e_B(x + I_K) = e(x) \qquad (x \in K_+[G]) \quad .$$

This completes the proof.

To show that B_K is a Hopf algebra we now only need a K-algebra morphism $s_B:B_K \to B_K$, called an 'antipode', satisfying

2.5c

$$m_B(1_B \otimes s_B)\Delta_B = m_B(s_B\otimes 1_B)\Delta_B = i_B e_B$$

where $m_B : B_K \otimes B_K \to B_K$ is multiplication, $1_B : B_K \to B_K$ is the identity map and $i_B : K \to B_K$ is the natural inclusion.

Define a K-algebra morphism $s : K_+[G] \to K_+[G]$ by extending multiplicatively and linearly to all of $K_+[G]$.

<u>2.5d</u>

$$s(c_{\mu\nu}) = c_{\overline{\nu\mu}} \qquad (\mu,\nu \in \underline{n})$$

$$s(\mathbb{1}) = \mathbb{1}$$

where for $\mu \in \underline{n}$, $\bar{\mu} = n+1-\mu$ and $\mathbb{1}$ is the identity of $K_+[G]$.

<u>2.5e</u> <u>Lemma</u>

We have $s(I_K) = I_K$.

<u>Proof</u>

Let $\alpha,\beta \in \underline{n}$, then

$$s(F^G_{\alpha\beta}) = s\Big(\big(\sum_{\mu\in\underline{n}} c_{\mu\alpha} c_{\bar{\mu}\beta}\big) - \delta_{\alpha,\bar{\beta}}.1\Big)$$

$$= \big(\sum_{\mu\in\underline{n}} c_{\overline{\alpha\mu}} c_{\bar{\beta}\mu}\big) - \delta_{\alpha,\bar{\beta}}.1$$

$$= H^G_{\overline{\alpha\beta}} .$$

Since I_K is generated as an ideal of $K_+[G]$ by $\{F^G_{\alpha\beta}, H^G_{\alpha\beta} : \alpha,\beta \in \underline{n}\}$ and s^2 is the identity map it follows that $s(I_K) = I_K$.

By (2.5e) we can define a K-algebra morphism

<u>2.5f</u>

$$s_B : B_K \to B_K$$

by $s_B(x+I_K) := s(x) + I_K \quad (x \in K_+[G])$.

For $\mu,\nu \in \underline{n}$ write $\bar{c}_{\mu\nu}$ for the image of $c_{\mu\nu} \in K_+[G]$ in B_K, then $s_B(\bar{c}_{\mu\nu}) = \bar{c}_{\bar{\nu}\bar{\mu}}$. Consider

$$m_B(1_B \otimes s_B)\Delta_B(\bar{c}_{\alpha\beta}) = m_B(1_B \otimes s_B)(\sum_\rho \bar{c}_{\alpha\rho} \otimes \bar{c}_{\rho\beta})$$

$$= \sum_\rho \bar{c}_{\alpha\rho}\bar{c}_{\bar{\beta}\bar{\rho}} \ .$$

This is the $(\alpha,\bar{\beta})^{th}$ coefficient of the matrix $\bar{C}J_n\bar{C}^t$ where $\bar{C} = (\bar{c}_{\mu\nu})_{\mu,\nu\in\underline{n}}$. Now, $H^G_{\alpha\bar{\beta}}$ is the $(\alpha,\bar{\beta})^{th}$ coefficient of $CJ_nC^t - J_n$ (2.1e) therefore since every $H^G_{\alpha\bar{\beta}}$ is in I_K we have $\bar{C}J_n\bar{C}^t = J_n$. Hence

$$m_B(1_B \otimes s_B)\Delta_B(\bar{c}_{\alpha\beta}) = \delta_{\alpha,\beta}\cdot 1_B$$

where 1_B is the identity element of B_K.

Similarly, using the $F^G_{\alpha\beta}$, we can show

$$m_B(s_B \otimes 1_B)\Delta_B(\bar{c}_{\alpha\beta}) = \delta_{\alpha,\beta}\cdot 1_B \ .$$

Finally, $i_B e_B(\bar{c}_{\alpha\beta}) = \delta_{\alpha\beta}(1 + I_K) = \delta_{\alpha\beta}\cdot 1_B$. Hence, (2.5c) holds and $s_B : B_K \to B_K$ is an antipode.

<u>2.5g</u> B_K is a Hopf algebra with antipode $s_B : B_K \to B_K$

Let H be any K-Hopf algebra. By (0.2d) we can give $H^* = \mathrm{Hom}_K(H,K)$ a K-algebra structure. Denote by G_H the K-subalgebra of H^* consisting of K-algebra morphisms $H \to K$. Then G_H is a group since if $a \in G_H$ then $as_H : H \to K$ is an element of G_H, where $s_H : H \to H$ is the antipode of H, and $a.(as_H) = (as_H).a = e_H \in G_H$ the counit of H which is the identity of H^* and hence of G_H. For each $a \in G_H$ denote the K-algebra automorphism $m_H(1_H \otimes a)\Delta_H : H \to H$ by ρ_a where m_H, Δ_H and 1_H are the multiplication, comultiplication and identity maps of H respectively. We state a lemma concerning these automorphisms whose proof can be found in [B, §4.2].

2.5h Lemma

Let K be an algebraically closed field and H a K-Hopf algebra which is finitely generated and commutative as a K-algebra. Take any $x \in H$, such that x is not in the nilradical of H, then the ideal of H generated by the elements $\rho_a(x)$ $(a \in G_H)$ is the whole of H.

We can now prove that B_K is reduced using an argument similar to that of Borel [B, §4.6].

2.5i Proposition

The K-algebra B_K is reduced.

Proof

Assume first that K is algebraically closed, then B_K satisfies the hypothesis of (2.5h). Let $N \trianglelefteq B_K$ be the nilradical of B_K, then N consists of all the nilpotent elements of B_K. Since $B_K[d_B^{-1}]$ is reduced (2.5a) the image of N in this algebra must be zero. Let $x_1, x_2, \ldots, x_t$ be ideal generators of N, then each $x_i/1 \in B_K[d_B^{-1}]$ is zero. Hence, for each $i \in \underline{t}$ there is some $s_i \in \underline{Z}_{>0}$ such that $x_i d_B^{s_i} = 0$ in B_K. Let $s = \max(s_i)$ then $N.d_B^s = 0$.

Now, for each $a \in G_B$, $\rho_a : B_K \to B_K$ is an automorphism so that $\rho_a(N) = N$. Moreover $N.d_B^s = 0$ implies $N.Y = 0$ where Y is the ideal of B_K generated by $\rho_a(d_B^s)$ $(a \in G_B)$. But since $d_B^s \notin N$ (else the algebra $B_K[d_B^{-1}] = (0)$ contradicting (2.4h)) $Y = A$ by (2.5h) and then $N = (0)$. This proves the proposition for K algebraically closed.

Suppose now that K is any infinite field (whose characteristic is not two) and denote by $\bar{K}$ its algebraic closure. It is easy to see that $B_{\bar{K}}$ is isomorphic as a $\bar{K}$-algebra to $B_K \otimes_K \bar{K}$. If $x \in B_K$ is nilpotent then $x \otimes 1$ is nilpotent and hence zero since $B_{\bar{K}}$ is reduced. Thus $x = 0$ in B_K and we are finished.

2.6 The kernel of $\Psi_K : K_+[G] \to K[\Gamma]$

Using the results of the previous sections, we have the following commutative diagram.

2.6a

$$
\begin{tikzcd}
 & J_K \arrow[r] \arrow[d] & K_+[G] \arrow[r, two heads, "\Psi_K"] \arrow[d] & K[\Gamma] \arrow[d] & \\
I_K[d^{-1}] \arrow[r, phantom, "="] & J_K[d^{-1}] \arrow[r] & K_+[G][d^{-1}] \arrow[r, two heads, "\Psi_K^{\Omega}"] & K[\Gamma][d_\Gamma^{-1}] & B_K[d_B^{-1}] \arrow[l, "\Phi"'] \\
I_K \arrow[u] \arrow[rr] & & K_+[G] \arrow[u] \arrow[r, two heads, "\theta"] & B_K \arrow[ur, "i"'] &
\end{tikzcd}
$$

The middle row of maps and identities come from (2.4 b,g,h). The top and bottom rows of vertical maps (excluding i) are all monomorphisms since $d \in K_+[G]$ and $d_\Gamma \in K[\Gamma]$ are non zero divisors (1.2e), (2.4c). As always we consider them as inclusions where convenient.

Finally, $i : B_K \to B_K[d_B^{-1}]$ is the natural map $b \mapsto b/1$.

2.6b Lemma

We have $\Phi i \theta = \Psi_K : K_+[G] \to K[\Gamma]$.

Proof

This follows straight from the definition of Φ (2.4h).

Assume K is *algebraically closed* in (2.6c,d).

2.6c Lemma

Every K-algebra map $a : K_+[G] \to K$ is evaluation at some unique $g \in M_{n,K}$, namely $g = (a(c_{\mu\nu}))_{\mu,\nu\in\underline{n}}$. Moreover $g \in \Gamma_K$ iff $F^G_{\alpha\beta}(g) = 0$ for all $\alpha,\beta \in \underline{n}$.

Proof

If we identify $K_+[G]$ with the coordinate ring of the algebraic variety $M_{n,K}$ (Remark, p.6), then the Nullestellensatz implies that a is evaluation at some element of $M_{n,K}$. Finally, $g \in \Gamma_K$ iff $g^tJg-J = 0$, but the $(\alpha,\beta)^{th}$ coefficient of this matrix is $F^G_{\alpha\beta}(g)$ and we are finished.

2.6d Lemma

Let $a \in G_B$, then there is an element $a' \in G_{K[\Gamma]} := K\text{-alg}(K[\Gamma],K)$ such that $a'\Phi i = a$.

Proof

Let $b = a\theta \in K\text{-alg}(K_+[G],K)$, then $b(F^G_{\alpha\beta}) = 0$ since

$F^G_{\alpha,\beta} \in \ker\theta = I_K \quad (\alpha,\beta\in\underline{n})$. By (2.6c) b is evaluation at some unique $g \in \Gamma_K$. Let $a' \in G_{K[\Gamma]}$ be evaluation at this $g \in \Gamma_K$, then

$$a'\psi_K = a\theta : K_+[G] \to K .$$

But, by (2.6b), $\psi_K = \phi i\theta$ and since θ is surjective

$$a'\phi i = a : B_K \to K ,$$

as required.

We can now prove

2.6e Theorem

Let K be an infinite field, characteristic not two. Then $I_K = J_K$.

Proof

It is enough to show that $i : B_K \to B_K[d_B^{-1}]$ is injective, since then $I_K = \ker\theta = \ker\phi i\theta = \ker\psi_K = J_K$ by (2.6b).

Assume first that K is algebraically closed. Let $a \in G_B$, then by (2.6d) there exists an $a' \in G_{K[\Gamma]}$ such that $a = a'\phi i$. Then

$$a(\ker\phi i) = a'\phi i(\ker\phi i) = 0$$

so that $\ker\phi_i \subseteq \bigcap_{a\in G_B} \ker a$. But, since B_K is a finitely generated K-algebra by the Nullstellensatz this intersection is the nilradical of B_K . Hence, by (2.5i), $\ker\phi i = \{0\}$ and i is therefore injective. This proves the theorem in the case K algebraically closed.

Now, let K be any infinite field (char $K \neq 2$) and denote by $\overline{K}$ its algebraic closure. There exists a K-algebra monomorphism $w: B_K \to B_{\overline{K}}$ given by

$$w(c_{\mu\nu}^{G(K)} + I_K) = c_{\mu\nu}^{G(\overline{K})} + I_{\overline{K}} \qquad (\mu,\nu \in \underline{n})$$

where $c_{\mu\nu}^{G(K)}$ and $c_{\mu\nu}^{G(\overline{K})}$ denote the $(\mu,\nu)^{th}$ coefficient function in $K_+[G_K]$ and $\overline{K}_+[G_{\overline{K}}]$ respectively.

Now, w takes d_{B_K} to $w(d_{B_K}) = d_{B_{\overline{K}}}$ and since $i_{\overline{K}} : B_{\overline{K}} \to B_{\overline{K}}[d_{B_{\overline{K}}}^{-1}]$ was shown above to be injective, $d_{B_{\overline{K}}}$ is not a zero divisor. Hence, using w, d_{B_K} is not a zero divisor in B_K and therefore $i : B_K \to B_K[d_{B_K}^{-1}]$ is injective implying that $I_K = J_K$.

This completes the proof of the theorem.

2.6f Remark

In the case characteristic of K equal to zero this theorem was proved by Weyl [W, Theorem 5.2c] using classical invariant theory.

2.6g Remark

It is now easy to see that $K[\Gamma]$ is a K-Hopf algebra with antipode $s_\Gamma : K[\Gamma] \to K[\Gamma]$ such that $s_\Gamma(c_{\mu\nu}^{\Gamma}) = c_{\nu\mu}^{\Gamma}$ $(\mu,\nu \in \underline{n})$.

§3. Modular theory

3.1 The K-algebra $K \otimes Z[\Gamma_Q]$

For any infinite field K we have the s.e.s.

3.1a

$$J_K \xrightarrow{\text{inc}} K_+[G_K] \xrightarrow{\Psi_K} K[\Gamma_K]$$

where Ψ_K is the restriction map.

When $K = Q$, by further restriction, we have the s.e.s.

3.1b

$$J_Z \xrightarrow{\text{inc}} Z_+[G_Q] \xrightarrow{\Psi_Q} Z[\Gamma_Q]$$

where $Z_+[G_Q]$ is the subring of $Q_+[G_Q]$ Z-spanned by monomials in the coordinate functions and $J_Z = J_Q \cap Z_+[G_Q]$.

Since $Z[\Gamma_Q] = \Psi_Q(Z_+[G_Q])$ is torsion free as a Z-module it is a flat Z-module, [R, p.54] and therefore if N is any Z-module we have the s.e.s.

3.1c

$$N\otimes_Z J_Z \xrightarrow{\text{inc}} N\otimes_Z Z_+[G_Q] \xrightarrow{1\otimes\Psi_Q} N\otimes_Z Z[\Gamma_Q] \ .$$

In particular we shall be interested in the case when N is an infinite field, whose characteristic may be non zero, and the relation between (3.1a) and (3.1c). We shall prove there is an isomorphism of K-algebras (when $\mathrm{char} K \neq 2$)

$$K \otimes_Z Z[\Gamma_Q] \to K[\Gamma_K]$$

$$1 \otimes c_{\mu\nu}^{\Gamma(Q)} \mapsto c_{\mu\nu}^{\Gamma(K)} \qquad (\mu,\nu \in \underline{n}) .$$

This is one of the most important applications of Chevalley's theorem. Borel [B, §4] proves the corresponding result for all connected semisimple Chevalley groups, here our proof is a corollary of our knowledge of the kernel J_K .

For any infinite field K , $K_+[G_K]$ is a free polynomial ring over K on the generators $\{c_{\mu\nu}^{G(K)} : \mu,\nu \in \underline{n}\}$ and by definition $Z_+[G_Q]$ is a free polynomial ring over Z on the generators $\{c_{\mu\nu}^{G(Q)} : \mu,\nu \in \underline{n}\}$. Hence there is a natural isomorphism of K-algebras.

3.1d

$$K \otimes_Z Z_+[G_Q] \to K_+[G_K]$$

$$1 \otimes c_{\mu\nu}^{G(Q)} \mapsto c_{\mu\nu}^{G(K)} \qquad (\mu,\nu \in \underline{n}) .$$

Henceforth, let K be of *characteristic not two.*

Let Z^* denote the subring of Q generated by Z and $\frac{1}{2}$, and define $Z^*[U_{\Gamma(Q)}]$ to be the Z^*-subalgebra of $Q[U_{\Gamma(Q)}]$ generated by $\{c_{\mu\nu}^{U_{\Gamma(Q)}} : \mu,\nu \in \underline{n}\}$. By (1.31) this is freely generated by $\{c_{\mu\nu}^{U_{\Gamma(Q)}} : (\mu,\nu) \in R_\ell\}$

and it follows that the map

3.1e

$$K \otimes_{Z^*} Z^*[U_{\Gamma(Q)}] \to K[U_{\Gamma(K)}]$$

$$1 \otimes c_{\mu\nu}^{U_{\Gamma(Q)}} \mapsto c_{\mu\nu}^{U_{\Gamma(K)}} \qquad (\mu,\nu\in\underline{n})$$

is an isomorphism of K-algebras since by the proof of (1.3c) $K[U_{\Gamma(K)}]$ is freely generated by $\{c_{\mu\nu}^{U_{\Gamma(K)}} : (\mu,\nu) \in R_\ell\}$ in such a way that if, for any $\mu,\nu\in\underline{n}$,

$$c_{\mu\nu}^{U_{\Gamma(Q)}} = \sum_r \lambda(r) \prod_{(\alpha,\beta)\in R_\ell} (c_{\alpha\beta}^{U_{\Gamma(Q)}})^{r_{\alpha\beta}} \qquad (\lambda(r)\in Z^*)$$

then

$$c_{\mu\nu}^{U_{\Gamma(K)}} = \sum_r \overline{\lambda}(r) \prod_{(\alpha,\beta)\in R_\ell} (c_{\alpha\beta}^{U_{\Gamma(K)}})^{r_{\alpha\beta}} \qquad (\overline{\lambda}(r)\in K)$$

where $r = (r_{\alpha\beta}) \in Z_{\geq 0}^{|R_\ell|}$ and $\overline{\lambda}(r) = \lambda(r).1_K$. (remember we assume char $K \neq 2$) .

Similarly, we have an isomorphism of K-algebras

3.1f

$$K \otimes_{Z^*} Z^*[U^-_{\Gamma(Q)}] \to K[U^-_{\Gamma(K)}]$$

$$1 \otimes c_{\mu\nu}^{U^-_{\Gamma(Q)}} \mapsto c_{\mu\nu}^{U^-_{\Gamma(K)}} \qquad (\mu,\nu\in\underline{n}) .$$

We have an isomorphism of K-algebras

3.1g

$$K \otimes_{Z^*} Z^*[T_{\Gamma(Q)}] \to K[T_{\Gamma(K)}]$$

$$1 \otimes \chi_\alpha^{\Gamma(Q)} \mapsto \chi_\alpha^{\Gamma(K)} \qquad (\alpha \in \Lambda(\Gamma))$$

where $Z^*[T_{\Gamma(Q)}]$ is defined to be the Z^*-span of $\{\chi_\alpha^{\Gamma(Q)} : \alpha \in \Lambda(\Gamma)\}$. This latter set is clearly a basis of $Z^*[T_{\Gamma(Q)}]$ since the characters are even linearly independent over Q, (1.3, p.32). Therefore, since $K[T_{\Gamma(K)}]$ also has a basis of characters $\{\chi_\alpha^{\Gamma(K)} : \alpha \in \Lambda(\Gamma)\}$ (3.1g) follows immediately.

Recall that for any K we have an inclusion

$$K[\Gamma] \xrightarrow{\phi^*_{\Gamma(K)}} K[U^-_\Gamma] \otimes K[T_\Gamma] \otimes K[U_\Gamma]$$

where $\phi^*_{\Gamma(K)}$ is the restriction to $K[\Gamma] \subset K[\Gamma][d_\Gamma^{-1}]$ of the Chevalley isomorphism $\phi^*_{\Gamma(K)}$ (1.3m), (remembering that $d_\Gamma \in K[\Gamma]$ is a non zero divisor (2.4c)).

3.1h

Let $K = Q$, by the proof of Chevalley's theorem (1.3m) it is clear that

$$\phi^*_{\Gamma(Q)}(Z^*[\Gamma_Q]) \subset Z^*[U^-_{\Gamma(Q)}] \otimes Z^*[T_{\Gamma(Q)}] \otimes Z^*[U_{\Gamma(Q)}]$$

in such a way that if, for $\mu, \nu \in \underline{n}$,

$$\phi^*_{\Gamma(Q)}(c^{\Gamma(Q)}_{\mu\nu}) = \sum_{r',\gamma,r} \lambda(r',\gamma,r) \prod_{(\alpha,\beta)\in R^-_\ell} (c^{U^-_{\Gamma(Q)}}_{\alpha\beta})^{r'_{\alpha\beta}} \chi^{\Gamma(Q)}_\gamma \prod_{(\beta,\alpha)\in R_\ell} (c^{U_{\Gamma(Q)}}_{\beta\alpha})^{r_{\beta\alpha}}$$

then

$$\phi^*_{\Gamma(K)}(c^{\Gamma(K)}_{\mu\nu}) = \sum_{r',\gamma,r} \overline{\lambda}(r',\gamma,r) \prod_{(\alpha,\beta)\in R^-_\ell} (c^{U^-_{\Gamma(K)}}_{\alpha\beta})^{r'_{\alpha\beta}} \chi^{\Gamma(K)}_\gamma \prod_{(\beta,\alpha)\in R_\ell} (c^{U_{\Gamma(K)}}_{\beta\alpha})^{r_{\beta\alpha}}$$

where $(r',\gamma,r) \in Z^{\ell^2}_{\geq 0} \times \Lambda(\Gamma) \times Z^{\ell^2}_{\geq 0}$, $\lambda(r',\gamma,r) \in Z^*$ and $\overline{\lambda}(r',\gamma,r) = \lambda(r',\gamma,r) \cdot 1_K$.

Hence, using (3.1 d,e,f,g,h) we have the following diagram

$$\begin{array}{ccc}
K \otimes Z^*[\Gamma_Q] & \xrightarrow{1\otimes\phi^*_{\Gamma(Q)}} & (K\otimes Z^*[U^-_{\Gamma(Q)}])\otimes(K\otimes Z^*[T_{\Gamma(Q)}])\otimes(K\otimes Z^*[U_{\Gamma(Q)}]) \\
\Big\downarrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle i} \\
K[\Gamma_K] & \xrightarrow{\phi^*_{\Gamma(K)}} & K[U^-_{\Gamma(K)}]\otimes K[T_{\Gamma(K)}]\otimes K[U_{\Gamma(K)}]
\end{array}$$

where $i = (3.1e) \otimes (3.1g) \otimes (3.1f)$.

Now, maps $i(1 \otimes \phi^*_{\Gamma(Q)})$ and $\phi^*_{\Gamma(K)}$ have the same image since $i(1 \otimes \phi^*_{\Gamma(Q)})(1 \otimes c^{\Gamma(Q)}_{\mu\nu}) = \phi^*_{\Gamma(K)}(c^{\Gamma(K)}_{\mu\nu})$ $(\mu,\nu\in \underline{n})$ by (3.1h), and $\phi^*_{\Gamma(K)}$ is a monomorphism so there exists a K-algebra epimorphism

$$j := \phi^{*-1}_{\Gamma(K)} \circ i \circ (1 \otimes \phi^*_{\Gamma(Q)})$$

with $j(1 \otimes c_{\mu\nu}^{\Gamma(Q)}) = c_{\mu\nu}^{\Gamma(K)} \quad (\mu,\nu \in \underline{n})$.

We now have the following commutative diagram

3.1j

$$\begin{tikzcd} 0 \arrow[r] & K\otimes J_Z \arrow[r, "inc"] \arrow[d] & K\otimes Z_+[G_Q] \arrow[r, "1\otimes\tau_Q"] \arrow[d, "\beta"] & K\otimes Z[\Gamma_Q] \arrow[r] \arrow[d, "j"] & 0 \\ 0 \arrow[r] & J_K \arrow[r, "inc"] & K_+[G_K] \arrow[r, two heads, "\Psi_K"] & K[\Gamma_K] \arrow[r] & 0 \end{tikzcd}$$

where, since it is clear that $K \otimes Z^*[\Gamma_Q]$ is naturally isomorphic to $K \otimes Z[\Gamma_Q]$, we denote by $j : K\otimes Z[\Gamma_Q] \to K[\Gamma_K]$ the composition of $j : K\otimes Z^*[\Gamma_Q] \to K[\Gamma_K]$ with this isomorphism.

It is now clear, from the cummutativity of the right hand square of (3.1j), that $\beta(K \otimes J_Z) \subseteq J_K$. But in fact we have:

3.1k Theorem

Let K be any infinite field not of characteristic two, then $\beta(K \otimes J_Z) \supseteq J_K$ and hence $j : K \otimes Z[\Gamma_Q] \to K[\Gamma_K]$ is an isomorphism of K-algebras.

Proof

Recall that for any such K , J_K is generated as an ideal of $K_+[G_K]$ by the elements $\{F_{\alpha\beta}^{G(K)}, H_{\alpha\beta}^{G(K)} : \alpha,\beta \in \underline{n}\}$ (2.6a). Clearly $J_Z = Z_+[G_Q] \cap J_Q$ contains each $F_{\alpha\beta}^{G(Q)}, H_{\alpha\beta}^{G(Q)}$ $(\alpha,\beta \in \underline{n})$, hence $K \otimes J_Z$ contains $1 \otimes F_{\alpha\beta}^{G(Q)}$, $1 \otimes H_{\alpha\beta}^{G(Q)}$ $(\alpha,\beta \in \underline{n})$. Now $\beta(1 \otimes F_{\alpha\beta}^{G(Q)}) = F_{\alpha\beta}^{G(K)}$, $\beta(1 \otimes H_{\alpha\beta}^{G(Q)}) = H_{\alpha\beta}^{G(K)}$ and it follows that $\beta(K \otimes J_Z) \supseteq J_K$ and we are finished.

3.1ℓ Corollary

The monomorphism of Z^* algebras

$$Z^*[\Gamma_Q] \xrightarrow{\phi^*_{\Gamma(Q)}} Z^*[U^-_{\Gamma(Q)}] \otimes Z^*[T_{\Gamma(Q)}] \otimes Z^*[U_{\Gamma(Q)}] \ ,$$

is split as a Z^* module map.

Proof

By the proof of Chevalley's theorem (1.2i, 1.3m) $\phi^*_{\Gamma(Q)}(Z^*[\Gamma_Q])$ is a Z^*-submodule of the Z^* module $A_{Z^*} = Z^*[U^-_{\Gamma(Q)}] \otimes Z^*[T_{\Gamma(Q)}] \otimes Z^*[U_{\Gamma(Q)}]$. As usual we identify $Z^*[\Gamma_Q]$ with its image under $\phi^*_{\Gamma(Q)}$.

Now, for $\phi^*_{\Gamma(Q)}$ to be split as a Z^* module map it is enough to show that $Z^*[\Gamma_Q]$ is pure in A_{Z^*} , that is if $f \in A_{Z^*}$, $m \in Z^*_{>0}$ with $mf \in Z^*[\Gamma_Q]$ then $f \in Z^*[\Gamma_Q]$.

Suppose there is some $f \in A_{Z^*}$ with $mf \in Z^*[\Gamma_Q]$. Choose m minimal so that $mf \in Z^*[\Gamma_Q]$, then if $m = 2^t$ $(t \in Z)$, $f \in Z^*[\Gamma_Q]$ so suppose $m \neq 2^t$ for any $t \in Z$. Let $p \neq 2$ be a prime dividing m then the image of mf in A_K under the map

$$\begin{aligned} A_{Z^*} &\to A_K \\ a &\to i(1 \otimes_{Z^*} a) \end{aligned}$$

is zero if char$K = p$, using (3.1i). Now, the kernel of

$$\begin{aligned} Z^*[\Gamma_Q] &\to K \otimes Z^*[\Gamma_Q] \\ x &\mapsto 1 \otimes_{Z^*} x \end{aligned}$$

is $p.Z^*[\Gamma_Q]$ since $Z^*[\Gamma_Q]$ is free and by (3.1k) $K \otimes Z^*[\Gamma_Q]$ is isomorphic as K algebra to $K[\Gamma] \subset A_K$. Then $i(1\otimes mf) = j(1\otimes mf) = 0$ in $K[\Gamma]$ implies $mf \in p.\, Z^*[\Gamma_Q]$. But then $m/p.f \in Z^*[\Gamma_Q]$ contradicting the minimality of m since $m/p \in Z^*$.

This completes the proof.

3.2 Modular reduction

Theorem (3.1k) allows us to 'reduce modulo p' modules in $M_Q(\Gamma)$, the category of finite dimensional polynomial Γ_Q-modules, as follows:

Let $V_Q \in M_Q(\Gamma)$ and $V_Z \subseteq V_Q$ be a free Z-submodule of V_Q with Z-basis $\{v_1, v_2, \ldots, v_m\}$. Then V_Z is called an 'admissible Z-form' of V_Q if $\{v_1, v_2, \ldots, v_m\}$ is also a Q basis of V_Q and if, for $b \in \underline{m}$, $g \in \Gamma_Q$

$$g.v_b = \sum_{a \in \underline{m}} p_{ab}(g) v_a$$

has $p_{ab} \in Z[\Gamma_Q]$ $(a, b \in \underline{m})$.

Let K be any infinite field (char$K \neq 2$) then the K-space $K \otimes_Z V_Z$ has a K-basis $\{\bar{v}_a := 1\otimes v_a : a \in \underline{m}\}$ and we give it a $K\Gamma_K$ module structure by defining

3.2a

$$g.\bar{v}_b = \sum_{a \in \underline{m}} \bar{p}_{ab}(g) \bar{v}_a \qquad (b \in \underline{m},\ g \in \Gamma_K)$$

where $\overline{p}_{ab} = j(1 \otimes p_{ab})$ $(a,b \in \underline{m})$.

Thus $K \otimes_Z V_Z \in M_K(\Gamma)$.

3.2b Remark

It is a consequence of the isomorphism of (3.1k) that every Γ_Q-module in $M_Q(\Gamma)$ has an admissible Z-form, for details see for example [G',3.2], [Se].

§4. The Schur algebras of $O_{2\ell+1}(Q)$

Throughout this chapter K will as usual denote an infinite field whose characteristic is not two.

4.1 The coalgebras $K_r[\Gamma]$

Recall that for $r \geq 0$ we have defined $K_r[G]$ to be a subcoalgebra of $K_+[G]$ consisting of all polynomials of degree $\leq r$ in the coordinate functions $c_{\mu\nu}$ $(\mu,\nu \in \underline{n})$ and $K_r[\Gamma]$ to be the cosubalgebra $\Psi_K(K_r[G])$ of $K[\Gamma]$ where $\Psi_K : K_+[G] \to K[\Gamma]$ is the restriction map; (0.1j), (0.2a), (0.3, p.11). As in (0.3) we write

$$\Psi_{r,K} : K_r[G] \to K_r[\Gamma]$$

$$c_{\mu\nu} \mapsto c_{\mu\nu}^{\Gamma(K)} \qquad (\mu,\nu \in \underline{n})$$

for the restriction of Ψ_K to $K_r[G]$. Let $J_{r,K}$ be the kernel of $\Psi_{r,K}$; then it is clear that $J_{r,K} = J_K \cap K_r[G]$ where J_K is the kernel of Ψ_K (for which a generating set has been found, (2.6e)).

4.1a

For $r \geq 2$ define $I_{r,K}$ to be the subspace of $K_r[G]$ consisting of the linear span of the elements:

$$F_{\alpha\beta}^{G(K)} c_{ij} \;,\; H_{\alpha\beta}^{G(K)} c_{ij} \qquad (\alpha,\beta \in \underline{n} \;,\; (i,j) \in T_{r-2}(n))$$

and let $I_{0,K} = I_{1,K} = \{0\}$.

By the proof of (2.5b) both $\Delta(F_{\alpha\beta}^{G(K)}c_{ij})$ and $\Delta(H_{\alpha\beta}^{G(K)}c_{ij})$ are elements of $I_{r,K} \otimes K_r[G] + K_r[G] \otimes I_{r,K}$ and $e(F_{\alpha\beta}^{G(K)}c_{ij}) = e(H_{\alpha\beta}^{G(K)}c_{ij}) = 0$ where Δ is the comultiplication and e the counit of $K_+[G]$, both K-algebra maps. Thus $I_{r,K}$ is a coideal of $K_r[G]$.

Clearly $I_{r,K} \subseteq J_{r,K}$. In (4.6) we shall show that $I_{r,Q} = J_{r,Q}$ using the techniques of Weyl [W,§5]. In the process we show also that the Schur algebras $S_{r,Q}(\Gamma)$ are semisimple so that all its modules ie. all the modules in $M_{r,Q}(\Gamma)$, are completely reducible, [W,5.7G].

Before proceeding we prove the following lemma which will be required later.

<u>4.1b</u> <u>Lemma</u>

The K-algebra automorphism

$$\begin{aligned} s : K_+[G] &\longrightarrow K_+[G] \\ c_{\mu\nu} &\longmapsto c_{\overline{\nu\mu}} \qquad (\mu,\nu \in \underline{n}) \\ \mathbb{D} &\longmapsto \mathbb{D} \end{aligned}$$

defined in (2.5d), maps $K^r[G]$ into $K^r[G]$ and induces an anti-automorphism

$$\begin{aligned} s^* : S_K^r(G) &\longrightarrow S_K^r(G) \\ \zeta_{ij} &\longmapsto \zeta_{\overline{ji}} \qquad (i,j \in I(n,r)) \end{aligned}$$

where, for $h = (h_1,h_2,\ldots,h_r) \in I(n,r)$ we define $\bar{h}$ to be $(\bar{h}_1,\bar{h}_2,\ldots,\bar{h}_r) \in I(n,r)$.

Further, the bilinear form $B:F_K \times E_K \to K$, yielding the orthogonal group Γ_K, is 's^* contravariant', ie.

$$B^r((s^*\phi)(x),y) = B^r(x,\phi(y))$$

for all $x, y \in E_K^r$ and $\phi \in S_K^r(G)$ where $B^r : E_K^r \times E_K^r \to K$ is the non singular bilinear form on E_K^r induced by B, ie. for $r > 0$

$$B^r(e_i,e_j) = \prod_{\rho\in \underline{r}} B(e_{i_\rho},e_{j_\rho}) \quad (i,j\in I(n,r)),$$

and $B^0(\lambda,\lambda') = \lambda\lambda' \in K \quad (\lambda,\lambda'\in K)$.

Proof

Recall that $K^r[G]$ is the subcoalgebra of $K_+[G]$ consisting of all polynomials in the $c_{\mu\nu}$ $(\mu,\nu\in\underline{n})$ of degree r, then it is clear that $s(K^r[G]) = K^r[G]$.

For $\phi\in S_K^r(G)$ define $s^*(\phi)\in S_K^r(G)$ to be the composition $\phi\circ s$. If $r = 0$ the lemma is obvious. Let $r > 0$, then s^* is well defined, since $(i,j) \sim (h,k)$ iff $(\bar{j},\bar{i}) \sim (\bar{k},\bar{h})$ $(i,j,h,k\in I(n,r))$ and hence if $\xi_{ij} = \xi_{hk}$ then $s^*(\xi_{ij}) = \xi_{\bar{j}\bar{i}} = \xi_{\bar{k}\bar{h}} = s^*(\xi_{hk})$.

Clearly s^* is a linear bijection and for $\phi, \phi' \in S^r(G)$, $i,j \in I(n,r)$ we have:

$$(s^*\phi\phi')(c_{ij}) = \phi\phi'(c_{\bar{j}\bar{i}})$$

$$= \sum_{h\varepsilon I(n,r)} \phi(c_{\bar{j}h})\phi'(c_{h\bar{i}})$$

$$= \sum_{h\varepsilon I(n,r)} \phi(s(c_{hj}))\phi'(s(c_{ih}))$$

$$= \phi'\phi(s(c_{ij})) = (s^*\phi')(s^*\phi)(c_{ij})$$

Hence $s^*(\phi\phi') = (s^*\phi')(s^*\phi)$ for all $\phi,\phi' \varepsilon S_K^r(G)$. Also $s^*(\varepsilon) = \varepsilon$, where $\varepsilon = \sum_{i\varepsilon I(n,r)} \xi_{ii}$, the identity of $S_K^r(G)$.

Hence s^* is an antiautomorphism of $S_K^r(G)$. Finally, for $\phi\varepsilon S_K^r(G)$, $i,j\varepsilon I(n,r)$,

$$B^r((s^*\phi)e_i,e_j) = \sum_{h\varepsilon I(n,r)} s^*\phi(c_{hi})\ B^r(e_h,e_j)$$

$$= s^*\phi(c_{\bar{j}i}) = \phi(c_{\bar{i}j})$$

and

$$B^r(e_i,\phi e_j) = \sum_{h\varepsilon I(n,r)} \phi(c_{hj})\ B^r(e_i,e_h)$$

$$= \phi(c_{\bar{i}j}) .$$

Thus $B^r(s^*\phi(x),y) = B^r(x,\phi(y))$ for all $x,y \varepsilon E_K^r$ by linearity.

4.2 The KG-module E_K^r

Recall that G acts on the left of E_K^r by extending linearly to the whole of E_K^r the action:

$$g.e_j = \sum_{i\in I(n,r)} g_{ij}.e_i \qquad (g\in G, j\in I(n,r))$$

(see (0.1a,b)).

We also have an action of the symmetric group $G(r)$ on the right of E_K^r by extending linearly to the whole of E_K^r the action:

$$e_i.\pi = e_{i\pi} \qquad (i \in I(n,r),\ \pi \in G(r))$$

where $\qquad i\pi = (i_{\pi(1)}, i_{\pi(2)}, \ldots, i_{\pi(r)})$, (0.1d).

4.2a

For $g \in G$, $\pi \in G(r)$ and $i \in I(n,r)$ it is easy to see that $g(e_i.\pi) = (ge_i).\pi$. Thus the actions of G and $G(r)$ on E_K^r commute. Using (0.2g) it follows that the action of the Schur algebra $S_K^r(G)$ on E_K^r also commutes with this action of $G(r)$ and therefore if

$$\rho : S_K^r(G) \to \mathrm{End}_K(E_K^r)$$

is the representation afforded by E_K^r as a $S_K^r(G)$-module then $\rho(s_K^r(G)) \subseteq \mathrm{End}_{KG(r)}(E_K^r)$. In fact ρ is a monomorphism onto $\mathrm{End}_{KG(r)}(E_K^r)$ and hence $S_K^r(G) \cong \mathrm{End}_{KG(r)}(E_K^r)$; see [G, 2.6c].

4.3 The Γ-maps β^{r}_{ab} , $\beta^{r,+}_{ab}$

Throughout this section $B:E_K \times E_K \to K$ can be any non-singular bilinear form, as usual $\Gamma_B = \{g \in G : g^t\underline{B}g = \underline{B}\}$.

Our first move is to introduce some notation.

4.3a

For an integer $r > 0$ define $J(r) := \{(a,b) \in \underline{r} \times \underline{r} : a \neq b\}$ and $J_o(r) := \{(a,b) \in J(r) : a < b\}$. For $r \geq 2$ and $(a,b) \in J(r)$ we define $[a,b] : \underline{r-2} \to \underline{r}$ to be the unique order preserving map such that $\mathrm{Im}[a,b] = \underline{r} \backslash \{a,b\}$. For example, let $r = 5$, $a = 2$, $b = 4$ then $[2,4](1) = 1$, $[2,4](2) = 3$ and $[2,4](3) = 5$.

As an easy consequence of the definition of $[a,b]$ we have:

4.3b

Let $r \geq 2$ and $\pi \in G(r)$, then given any $(a,b) \in J(r)$ there exists a unique $\pi_{a,b} \in G(r-2)$ such that the following diagram commutes:

$$\begin{tikzcd}
\underline{r-2} \arrow[r, "{[a,b]}"] \arrow[d, "\pi_{ab}"'] & \underline{r} \arrow[d, "\pi"] \\
\underline{r-2} \arrow[r, "{[\pi(a),\pi(b)]}"] & \underline{r}
\end{tikzcd}$$

4.3c

For any integer $n > 0$ we can identify $i \in I(n,r)$ with a function $i : \underline{r} \to \underline{n}$ (and vice versa) by defining $i(\mu) = i_\mu$ $(\mu \in \underline{r})$. Then, for

$r \geq 2$, $(a,b) \in J(r)$ we write $i[a,b] \in I(n,r-2)$ for the composition of $i : \underline{r} \to \underline{n}$ with $[a,b] : \underline{r-2} \to \underline{r}$. Thus, if as above $r = 5$, $a = 2$, $b = 4$ then $(i_1,i_2,i_3,i_4,i_5)[2,4] = (i_1,i_3,i_5)$.

4.3d

As in (4.1a) given a non singular bilinear form $B : E_K \times E_K \to K$ for any $r \geq 0$ we can define a non singular bilinear form $B^r : E_K^r \times E_K^r \to K$ by

$$B^r(e_i,e_j) = \prod_{\rho\in\underline{r}} B(e_{i_\rho},e_{j_\rho})$$

for $r > 0$ and $B^0(\lambda,\lambda') := \lambda\lambda' \in K \quad (\lambda,\lambda' \in K)$.

We shall denote the basis of E_K^r dual to $\{e_i : i \in I(n,r)\}$ with respect to B^r by $\{f_i : i \in I(n,r)\}$ so that

$$B^r(e_i,f_j) = \delta_{ij} := \prod_{\rho\in\underline{r}} \delta_{i_\rho j_\rho} \qquad (i,j \in I(n,r))$$

where $\delta_{i_\rho j_\rho}$ is the Kronecker delta.

It follows that $f_j = f_{j_1} \otimes f_{j_2} \otimes \ldots \otimes f_{j_r}$ where $\{f_\mu\}_{\mu\in\underline{n}}$ is the basis of E_K dual to $\{e_\mu\}_{\mu\in\underline{n}}$.

For $r \geq 2$, $(a,b) \in J(r)$ we define

4.3e

$$B_{ab}^r : E_K^r \to E_K^{r-2}$$

by extending linearly to all of E_K^r .

4.3f

$$\beta_{ab}^{r}(e_i) = B(e_{i_a}, e_{i_b})e_{i[a,b]} \quad (i \in I(n,r)) .$$

Further, we define

4.3g

$$\beta_{ab}^{r,+} : E_K^{r-2} \to E_K^{r}$$

to be the map dual to β_{ab}^{r} with respect to the non singular bilinear form $B : E_K \times E_K \to K$, thus

$$B^{r}(\beta_{ab}^{r,+}(x),y) = B^{r-2}(x,\beta_{ab}(y)) \quad (x \in E_K^{r-2}, y \in E_K^{r}) .$$

It follows that

4.3h

$$\beta_{ab}^{r,+}(e_i) = \sum_{\mu,\nu \in \underline{n}} B(f_\mu,f_\nu)e_{i(a,b:\mu,\nu)} \quad (i \in I(n,r-2))$$

where $i(a,b:\mu,\nu)$ is the unique element $j \in I(n,r)$ such that $j[a,b] = i$ and $j_a = \mu$, $j_b = \nu$.

It is clear that β_{ab}^{r} and $\beta_{ab}^{r,+}$ are both $K\Gamma_B$ maps and we also have:

<u>4.3i</u> <u>Lemma</u> (i) $\beta^r_{ab}(x\pi) = \beta^r_{\pi(a)\pi(b)}(x).\pi_{ab}$

(ii) $\beta^{r,+}_{\pi(a)\pi(b)}(y\pi_{ab}) = \beta^{r,+}_{ab}(y).\pi$

for $x \in E^r_K$, $y \in E^{r-2}_K$, $\pi \in G(r)$, $(a,b) \in J(r)$.

<u>Proof</u>

(i) For $i \in I(n,r)$

$$\beta_{ab}(e_i.\pi) = B(e_{i_{\pi(a)}}, e_{i_{\pi(b)}})e_{i\pi[a,b]}$$

but by (4.3b) there is a unique $\pi_{ab} \in G(r-2)$ such that $\pi[a,b] = [\pi(a),\pi(b)]\pi_{ab}$. Hence

$$\beta^r_{ab}(e_i.\pi) = B(e_{i_{\pi(a)}}, e_{i_{\pi(b)}})e_{i[\pi(a),\pi(b)]\pi_{ab}}$$

$$= (B(e_{i_{\pi(a)}}, e_{i_{\pi(b)}})e_{i[\pi(a),\pi(b)]}).\pi_{ab}$$

$$= \beta^r_{\pi(a)\pi(b)}(e_i).\pi_{ab} .$$

(ii) Use (4.3h) and (i).

<u>4.4</u> <u>Traceless Tensors</u>

Throughout this section $B : E_K \times E_K \to K$ can be any non singular symmetric, or antisymmetric bilinear form, then $\beta^r_{ab} = \beta^r_{ba}$, $\beta^{r,+}_{ab} = \beta^{r,+}_{ba}$ (B symmetric) and $\beta^r_{ab} = -\beta^r_{ba}$, $\beta^{r,+}_{ab} = -\beta^{r,+}_{ba}$ (B antisymmetric).

In his book 'Classical Groups' H. Weyl describes when char $K = 0$ a subspace of E^r_K consisting of all 'traceless tensors'. We can generalise this as follows.

We define a $K\Gamma_B$-map

4.4a

$$\beta^r : E^r_K \to \bigoplus_{(a,b)\in J_o(r)} E^{r-2}_K$$

given by $\beta^r(x) = \bigoplus_{(a,b)\in J_o(r)} \beta^r_{ab}(x)$ $(x \in E^r_K)$ with $J_o(r)$ ordered lexicographically.

4.4b Example

Let $r = 3$, then $\beta^3(e_i) = (\beta^3_{12}(e_i), \beta^3_{13}(e_i), \beta^3_{23}(e_i))$ $(i \in I(n,3))$.

When no confusion should arise we will write $\bigoplus_{(a,b)\in J_o(r)}$ as $\bigoplus$.

4.4c

The space of 'traceless tensors' is defined to be the kernel $\ker_K \beta^r = \bigcap_{(a,b)\in J_o(r)} \ker_K \beta^r_{ab}$. Clearly this is a $K\Gamma_B$ submodule of E^r_K and by (4.3i) it is also a $KG(r)$ submodule of (the $KG(r)$-module) E^r_K .

The non singular bilinear form $B^{r-2} : E_K^{r-2} \times E_K^{r-2} \to K$ induces a non singular bilinear form $\oplus B^{r-2} : \oplus E_K^{r-2} \times \oplus E_K^{r-2} \to K$ with

$$\oplus B^{r-2} : (\ldots, x_{ab}, \ldots) \times (\ldots, y_{ab}, \ldots) \longrightarrow \sum_{(a,b) \in J_o(r)} B^{r-2}(x_{ab}, y_{ab})$$

for all $x_{ab}, y_{ab} \in E_K^{r-2}$ $((a,b) \in J_o(r))$.

We define the $K\Gamma_B$-map

4.4d

$$\beta^{r,+} : \oplus E_K^{r-2} \to E_K^r$$

to be that dual to $\beta^r : E_K^r \to \oplus E_K^{r-2}$ with respect to the form B^r on E_K^r and $\oplus B^{r-2}$ on $\oplus E_K^{r-2}$. It follows that

4.4e

$$\beta^{r,+}(\ldots, x_{ab}, \ldots) = \sum_{(a,b) \in J_o(r)} \beta_{ab}^{r,+}(x_{ab})$$

for all $x_{ab} \in E_K^{r-2}$ $((a,b) \in J_o(r))$.

Clearly the image of $\beta^{r,+}$, $\mathrm{Im}_K \beta^{r,+}$ is a $K\Gamma_B$ submodule of E_K^r and by (4.3i) it is also a $KG(r)$-submodule of (the $KG(r)$-module) E_K^r . We also have:

4.4f Lemma

$(\mathrm{Im}_K \beta^{r,+})^{\perp} = \ker_K \beta^r$ where the orthogonal complement is taken with respect to the form B^r on E_K^r.

Proof

Let $x \in E_K^r$. Then $x \in (\mathrm{Im}_K \beta^{r,+})^{\perp}$ iff $B^r(x, \beta^{r,+}(y)) = 0$, for all $y \in \oplus E_K^{r-2}$ iff $\oplus B^{r-2}(\beta^r(x), y) = 0$, for all $y \in \oplus E_K^{r-2}$ (since $\beta^{r,+}$ is dual to β^r) iff $\beta^r(x) = 0$ iff $x \in \ker_K \beta^r$.

4.4g Corollary

$$\dim_K(\mathrm{Im}_K \beta^{r,+}) + \dim_K(\ker_K \beta^r) = \dim_K E_K^r .$$

4.5 Decomposition of E_K^r in characteristic zero

Let $A : E_Q \times E_Q \to Q$ be the non singular symmetric bilinear form defined by

$$A(e_\mu, e_\nu) = \delta_{\mu,\nu} \qquad (\mu, \nu \in \underline{n})$$

and $\gamma^r : E_Q^r \to \oplus E_Q^{r-2}$, $\gamma^{r,+} : \oplus E_Q^{r-2} \to E_Q^r$ be the maps constructed using A as $\beta^r, \beta^{r,+}$ where constructed in (4.4) using B.

4.5a Remark

This is the form on E_Q^r used by Weyl and $\ker_Q \gamma^r$ is the space of traceless tensors [W,p.150].

We have the following two results concerning this form.

4.5b **Proposition** [W,5.6A]

We have for any $r \geq 2$

$$E^r_Q = \mathrm{Im}_Q\gamma^{r,+} \oplus \ker_Q\gamma^r \quad .$$

Proof

By (4.4g) we have

$$\dim_Q E^r_Q = \dim_Q \mathrm{Im}_Q\gamma^{r,+} + \dim_Q \ker_Q\gamma^r$$

so it is enough to check that the intersection of $\mathrm{Im}_Q\gamma^{r,+}$ with $\ker_Q\gamma^r$ is zero. Suppose $x = \Sigma\lambda_i e_i \in \mathrm{Im}_Q\gamma^{r,+} \cap \ker_Q\gamma^r$ $(\lambda_i \in Q, i \in I(n,r))$, then $A^r(x,x) = 0$ which implies that $\Sigma\lambda_i^2 = 0$ and then $\lambda_i = 0$ for all $i \in I(n,r)$. Thus $x = 0$ and we are finished.

We can extend this decomposition to the case $K = \mathbb{C}$, the complex numbers, or any other field of characteristic zero. Assume E_Q , E^r_Q and $\oplus E^{r-2}_Q$ are naturally embedded in $E_{\mathbb{C}}$, $E^r_{\mathbb{C}}$ and $\oplus E^{r-2}_{\mathbb{C}}$ and extend $A, A^r, \gamma^r, \gamma^r_{ab}, \gamma^{r,+}$ and $\gamma^{r,+}_{ab}$ to $E_{\mathbb{C}}, E^r_{\mathbb{C}}, \oplus E^{r-2}_{\mathbb{C}}$ and $E^{r-2}_{\mathbb{C}}$ as appropriate.

4.5c **Lemma**

(i) $\mathrm{Im}_{\mathbb{C}}\gamma^{r,+} = \mathbb{C}.\mathrm{Im}_Q\gamma^{r,+}$

(ii) $\ker_{\mathbb{C}}\gamma^r = \mathbb{C}.\ker_Q\gamma^r$.

and hence $E^r_{\mathbb{C}} = \mathrm{Im}_{\mathbb{C}}\gamma^{r,+} \oplus \ker_{\mathbb{C}}\gamma^r$.

Proof

(i) The set $\{\gamma^{r,+}_{ab}(e_i) : (a,b) \in J_0(r),\ i \in I(n,r-2)\}$ spans both $\mathrm{Im}_Q\gamma^{r,+}$ and $\mathrm{Im}_{\mathbb{C}}\gamma^{r,+}$.

(ii) The s.e.s.

$$0 \to \ker_Q\gamma^r \to E^r_Q \to \mathrm{Im}_Q\gamma^r \to 0$$

splits and so remains exact on tensoring with $\mathbb{C}$.

Let $B : E_{\mathbb{C}} \times E_{\mathbb{C}} \to \mathbb{C}$ be any nonsingular symmetric bilinear form, by elementary linear algebra we have a non singular map $P : E_{\mathbb{C}} \to E_{\mathbb{C}}$ such that

4.5d

$$B(Px,Py) = A(x,y) \qquad \forall\ x,y \in E_{\mathbb{C}} .$$

Note that in general it is not true that $P(E_Q) = E_Q$.

As a consequence we have:

4.5e Lemma

(i) $\mathrm{Im}_{\mathbb{C}}\beta^{r,+} = P^r(\mathrm{Im}_{\mathbb{C}}\gamma^{r,+})$.

(ii) $\ker_{\mathbb{C}}\beta^r = P^r(\ker_{\mathbb{C}}\gamma^r)$.

(iii) $E^r_{\mathbb{C}} = \mathrm{Im}_{\mathbb{C}}\beta^{r,+} \oplus \ker_{\mathbb{C}}\beta^r$.

where $P^r := P \otimes P \otimes \ldots \otimes P : E^r_{\mathbb{C}} \to E^r_{\mathbb{C}}$, (r-tensor product).

Proof

It is easy to show using (4.5d) that $\beta^r P^r = P^{r-2}\gamma^r$ and $\beta^{r,+}P^{r-2} = P^r\gamma^{r,+}$ where P^{r-2} acts on $\oplus E_C^{r-2}$ by $P^{r-2}(\ldots,x_{ab},\ldots) = (\ldots,P^{r-2}x_{ab},\ldots)$ $(x_{ab}\epsilon E_C^{r-2},(a,b) \in J_o(r))$.
Then (i) and (ii) are immediate and (iii) follows using (4.5c).

We can now return to the case $K = Q$. Since $\ker_Q\beta^r \subseteq \ker_{\mathbb{C}}\beta^r$ and $\mathrm{Im}_Q\beta^{r,+} \subseteq \mathrm{Im}_{\mathbb{C}}\beta^{r,+}$ we have $\ker_Q\beta^r \cap \mathrm{Im}_Q\beta^{r,+} = \{0\}$. Also, by (4.4g) $\dim_Q\ker_Q\beta^r + \dim_Q\mathrm{Im}_Q\beta^{r,+} = \dim_Q E_Q^r$ so that

4.5f

$$E_Q^r = \mathrm{Im}_Q\beta^{r,+} \oplus \ker_Q\beta^r .$$

4.6 Connection with the Schur algebras $S_{r,Q}(\Gamma)$

Let K be arbitrary not of characteristic two and $B : E_K \times E_K \to K$ be the non singular symmetric bilinear form yielding the odd orthogonal group Γ_K .

Consider the s.e.s.

4.6a

$$0 \to I_{r,K} \to K_r[G] \xrightarrow{\theta_{r,K}} B_{r,K} \to 0$$

where $I_{r,K}$ is the coideal defined in (4.1a), $B_{r,K}$ is the K-coalgebra $K_r[G]/I_{r,K}$ and $\theta_{r,K}$ is the natural epimorphism.

Then $\theta_{r,K}$ induces an injective morphism of K-algebras

$$S_{r,K} := B^*_{r,K} \to S_{r,K}(G)$$

where $B^*_{r,K} = \mathrm{Hom}_K(B_{r,K}, K)$ is given the algebra structure derived from the coalgebra structure of $B_{r,K}$ (0.2d) and $S_{r,K}(G)$ is the direct sum of the Schur algebras $S^{\rho}_K(G)$ $(0\leq\rho\leq r)$ (0.2h).

We identify $S_{r,K}$ with its image in $S_{r,K}(G)$, then

4.6b

$$S_{r,K} = \{\phi \in S_{r,K}(G) : \phi(I_{r,K}) = 0\} .$$

Now

4.6c

$$S_{r,K}(\Gamma) = \{\phi \in S_{r,K}(G) : \phi(J_{r,K}) = 0\}$$

and since $I_{r,K} \subseteq J_{r,K}$ it follows that $S_{r,K}(\Gamma) \subseteq S_{r,K}$.

Weyl [W,§4.2] used the complete reducibility of orthogonal transformations acting on E^r_Q to show that $S_{r,Q}$ is in fact equal to $S_{r,Q}(\Gamma)$.

This is also our aim but we shall deduce complete reducibility along the way.

We have the following characterisation of elements of $S_{r,K}$.

4.6d Proposition

Let $r \geq 2$, $\phi = (\phi^{(r)}, \phi^{(r-1)}, \ldots, \phi^{(0)}) \in S_{r,K}(G)$.
Then $\phi \in S_{r,K}$ if and only if

(i) $\phi^{(\rho)}\beta^{\rho,+} = \beta^{\rho,+}\phi^{(\rho-2)}$ for all $2 \leq \rho \leq r$

and

(ii) $\phi^{(\rho-2)}\beta^{\rho} = \beta^{\rho}\phi^{(\rho)}$ for all $2 \leq \rho \leq r$

where $\phi^{(\rho-2)}$ acts on $\oplus E_K^{\rho-2}$ by $\phi^{(\rho-2)}(\ldots,x_{ab},\ldots)$
$= (\ldots,\phi^{(\rho-2)}(x_{ab}),\ldots)$ $(x_{ab} \in E_K^{\rho-2}, (a,b) \in J_o(r))$.

Proof

We shall show (i) holds iff ϕ annihilates each
$H_{\mu\nu}^{G(K)}c_{ij}$ $(\mu,\nu \in \underline{n}$, $i,j \in I(n,\rho-2)$, $2 \leq \rho \leq r)$.

Now, $\phi^{(\rho)}\beta^{\rho,+} = \beta^{\rho,+}\phi^{(\rho-2)}$ iff $\phi^{(\rho)}\beta_{12}^{\rho,+} = \beta_{12}^{\rho,+}\phi^{(\rho-2)}$ by (4.3i(ii)) and the fact that the actions of $S_K^{\sigma}(G)$ and $G(\sigma)$ on E_K^{σ} commute for all $\sigma \geq 0$, (4.2a).

Let $i \in I(n,\rho-2)$, then

$$\phi^{(\rho)}\beta^{\rho,+}(e_i) = \phi^{(\rho)}\left(\sum_{\mu\in\underline{n}} e_\mu e_{\bar{\mu}} e_i\right) \tag{4.3h}$$

$$= \sum_{k\in I(n,\rho)} \left(\sum_{\mu\in\underline{n}} \phi(c_{k,i(1,2:\mu,\bar{\mu})})\right) e_k$$

and

$$\beta^{\rho,+}\phi^{(\rho-2)}(e_i) = \beta^{\rho,+}\left(\sum_{h\in I(n,\rho-2)} \phi(c_{hi})e_h\right)$$

$$= \sum_h \left(\sum_{\nu\in\underline{n}} \phi(c_{hi})\right) e_{h(1,2:\nu,\bar{\nu})} .$$

Now, by definition $k = h(1,2:\nu,\bar{\nu})$ iff $k[1,2] = h$ and $k_1 = \nu$, $k_2 = \bar{\nu}$ (cf. 4.3h). Thus, by comparing coefficients of the e_j $(j \in I(n,\rho-2))$ in the sums above we have

$$\phi^{(\rho)}\beta^{\rho,+}(e_i) = \beta^{\rho,+}\phi^{(\rho-2)}(e_i) \qquad (2 \leq \rho \leq r)$$

if and only if

$$\phi\left(\sum_{\mu\in\underline{n}} c_{k_1\mu}c_{k_2\bar{\mu}}\, c_{k[1,2]i}\right) = \delta_{k_1,\bar{k}_2}\, c_{k[1,2]i}$$

(for all $k \in I(n,\rho)$, $2 \leq \rho \leq r$)

if and only if

$$\phi(H^{G(K)}_{k_1k_2}\, c_{k[1,2]i}) = 0$$

(for all $k \in I(n,\rho)$, $2 \leq \rho \leq r$)

as required since as k runs through $I(n,\rho)$ k_1 and k_2 run through $\underline{n}$ and $k[1,2]$ runs through $I(n,\rho-2)$ $(2 \leq \rho \leq r)$.

Now, let $\bar{\phi} := s^*\phi = (s^*\phi^{(r)}, s^*\phi^{(r-2)}, \ldots, s^*\phi^{(0)}) \in S_{r,K}(G)$, (4.1b). Then

$$\phi^{(\rho-2)}\beta^{\rho} = \beta^{\rho}\phi^{(\rho)}$$

if and only if

$$\phi^{(\rho-2)}\beta_{12}^{\rho} = \beta_{12}^{\rho}\,\phi^{(\rho)} \qquad (4.3i(i))$$

if and only if

$$B^{\rho-2}(\phi^{(\rho-2)}\beta_{12}(x),y) = B^{\rho-2}(\beta_{12}^{\rho}\phi^{(\rho)}(x),y)$$

(for all $x \in E_K^{\rho}$, $y \in E_K^{\rho-2}$)

if and only if (using (4.1b) and (4.4e))

$$B^{\rho-2}(\beta_{12}^{\rho}(x),\bar{\phi}^{(\rho-2)}(y)) = B^{\rho-2}(\phi^{(\rho)}(x),\beta_{12}^{\rho,+}(y))$$

(for all $x \in E_K^{\rho}$, $y \in E_K^{\rho-2}$)

if and only if (using (4.1b) and (4.4e))

$$B^{\rho}(x,\beta_{12}^{\rho,+}\bar{\phi}^{(\rho-2)}(y)) = B^{\rho}(x,\bar{\phi}^{(\rho)}\beta_{12}^{\rho,+}(y))$$

(for all $x \in E_K^{\rho}$, $y \in E_K^{\rho-2}$)

if and only if

$$\beta_{12}^{\rho,+}\bar{\phi}^{(\rho-2)} = \bar{\phi}^{(\rho)}\beta_{12}^{\rho,+}$$

if and only if

$$\bar{\phi}^{(\rho)}(H_{\mu\nu}^{G(K)}\, c_{h,k}) = 0$$

for all $\mu,\nu \in \underline{n}$, $h,k \in I(n,\rho-2)$.

But $\bar{\phi}^{(\rho)}(H_{\mu\nu}^{G(K)}\, c_{hk}) = \phi^{(\rho)}(s(H_{\mu\nu}^{G(K)}\, c_{hk}))$

and $s(H_{\mu\nu}^{G(K)} c_{hk}) = F_{\mu\nu}^{-\underline{G}(K)}\, c_{\bar{k}\bar{h}}$ (cf. proof of (2.5e)).

Hence, (i) holds if and only if

$$\phi^{(\rho)}(F_{\mu\nu}^{G(K)}\, c_{hk}) = 0$$

for all $\mu,\nu \in \underline{n}$, $h,k \in I(n,\rho-2)$ and $2 \leq \rho \leq r$.

Since the set $\{F_{\mu\nu}^{G(K)}c_{hk}$, $H_{\mu\nu}^{G(K)}c_{hk}$: $\mu,\nu\in\underline{n}$ $h,k \in I(n,\rho-2)$ $2 \leq \rho \leq r\}$ spans $I_{r,K}$ (4.1a) we have shown that (i) and (ii) hold if and only if $\phi(I_{r,K}) = 0$ ie. if and only if $\phi \in S_{r,K}$ as required.

We shall also need

4.6e Lemma Weyl [W,p.152]

Let $\phi \in S_{r-2,K}$, then

$$\phi^{(r-2)}\beta^r\,\beta^{r,+} = \beta^r\,\beta^{r,+}\phi^{(r-2)} .$$

Proof

Weyl [W] proves this lemma in the case where the bilinear form is $A : E_Q \times E_Q \to Q$, however with slight modification the proof works for any symmetric bilinear form and infinite field.

We now consider the case $K = Q$.

Recall (4.4c) that $\ker_Q\beta^r$ is a $QG(r)$ submodule of E_Q^r and we may therefore consider the Q-subalgebra $\mathrm{End}_{QG(r)}(\ker_Q\beta^r)$ of $\mathrm{End}_Q(\ker_Q\beta^r)$. We have:

4.6f Proposition Weyl [W,p.152]

Let $r \geq 2$, then $S_{r,Q}$ is isomorphic as a Q algebra to $\mathrm{End}_{QG(r)}(\ker_Q\beta^r) \oplus S_{r-1,Q}$ such that restriction onto the second factor yields the Q-algebra epimorphism

$$S_{r,Q} \twoheadrightarrow S_{r-1,Q}$$

$$(\phi^{(r)},\phi^{(r-1)},\ldots,\phi^{(o)}) \to (\phi^{(r-1)},\phi^{(r-2)},\ldots,\phi^{(o)}) .$$

Proof

First, recall that $S_Q^r(G)$ acts faithfully on E_Q^r (cf. 4.2a) and hence the action of an element of $S_Q^r(G)$ on E_Q^r defines it uniquely.

We define a map

$$\kappa : \mathrm{End}_{QG(r)}(\ker_Q \beta^r) \oplus S_{r-1,Q} \to S_{r,Q}(G)$$

by $\kappa(\theta,\zeta) = \phi \in S_{r,Q}(G)$ where

$$\phi^{(\rho)} = \zeta^{(\rho)} \qquad 0 \leq \rho < r$$

and $\phi^{(r)}$ acts on E_Q^r as follows:

if $x \in E_Q^r$, use the decomposition (4.4f) to write $x = y + \beta^{r,+}(z)$ with $y \in \ker_Q \beta^r$, $z = (\ldots.,z_{ab},\ldots) \in \oplus E_Q^{r-2}$, then

$$\phi^{(r)}(x) = \phi^{(r)}(y+\beta^{r,+}(z)) := \theta(y) + \beta^{r,+}(\zeta^{(r-2)}(z))$$

where $\zeta^{(r-2)}(\ldots.,z_{ab},\ldots) = (\ldots,\zeta^{(r-2)}(z_{ab}),\ldots)$ as usual.

We must show κ is well defined since there are many choices of $z \in \oplus E_Q^{r-2}$ which yield the same image in E_Q^r under $\beta^{r,+}$. By linearity of $\beta^{r,+}$ and $\zeta^{(r-2)}$ it is enough to show that $\beta^{r,+}(z) = 0$ implies $\beta^{r,+}(\zeta^{(r-2)}(z)) = 0$. Further, since $\ker_Q\beta^r \cap \mathrm{Im}_Q\beta^{r,+} = \{0\}$ we have $\beta^{r,+}(z) = 0$ iff $\beta^r\,\beta^{r,+}(z) = 0$. Now, by (4.6e)

$$\zeta^{(r-2)}\beta^r\beta^{r,+}(z) = \beta^r\beta^{r,+}\zeta^{(r-2)}(z) \qquad (z \in \oplus E_Q^{r-2})$$

and hence if $\beta^{r,+}(z) = 0$

$$\zeta^{(r-2)}\beta^{r}(0) = 0 = \beta^{r}\beta^{r,+}\zeta^{(r-2)}(z)$$

and therefore $\beta^{r,+}\zeta^{(r-2)}(z) = 0$ as required.

By construction κ is injective and a morphism of Q-algebras, we show next that its image is in $S_{r,Q} \subseteq S_{r,Q}(G)$.

By (4.6d) if $\phi \in S_{r,Q}(G)$ then $\phi \in S_{r,Q}$ iff (i) and (ii) of (4.6d) hold. Now, if $\phi = \kappa(\theta,\zeta)$, then

$$\phi^{(\rho)}\beta^{\rho,+} = \beta^{\rho,+}\phi^{(\rho-2)} \qquad ..... \qquad (A)$$

and

$$\phi^{(\rho-2)}\beta^{\rho} = \beta^{\rho}\,\phi^{(\rho)} \qquad ..... \qquad (B)$$

for all $2 \leq \rho < r$ since by definition of κ , $\phi^{(\rho)} = \zeta^{(\rho)}$ $(2 \leq \rho < r)$ and $\zeta \in S_{r-1,Q}$. Also for $z \in \oplus E_Q^{r-2}$.

$$\phi^{(r)}\beta^{r,+}(z) := \beta^{r,+}(\zeta^{(r-2)}(z)) := \beta^{r,+}(\phi^{(r-2)}(z))$$

giving

$$\phi^{(r)}\beta^{r,+} = \beta^{r,+}\phi^{(r-2)} \qquad ..... \qquad (C)$$

and for $x \in E_Q^{r}$

$$\phi^{(r-2)}\beta^{r}(x) = \phi^{(r-2)}\beta^{r}(y+\beta^{r,+}(z)) = \beta^{r}\beta^{r,+}\phi^{(r-2)}(z)$$

by (4.6e) where $x = y + \beta^{r,+}(z)$ $(y \in \ker_Q\beta^r$, $z \in \oplus E_Q^{r-2})$. But
$\beta^{r,+}\phi^{(r-2)}(z) := \beta^{r,+}\zeta^{(r-2)}(z) := \phi^{(r)}\beta^{r,+}(z)$ and hence

$$\phi^{(r-2)}\beta^r(x) = \beta^r\phi^{(r)}\beta^{r,+}(z) = \beta^r\phi^{(r)}(y + \beta^{r,+}(z))$$

since $\phi^{(r)}(y) = \theta(y) \in \ker_Q\beta^r$. Hence

$$\phi^{(r-2)}\beta^r = \beta^r\phi^{(r)} . \qquad \ldots\ldots \qquad (D)$$

We have now shown (A,B,C,D) that $\phi = \kappa(\theta,\zeta)$ satisfies (i) and (ii) of (4.6d), thus $\phi \in S_{r,Q}$ as required. For κ to be an isomorphism of Q-algebras, it remains only to show that it is surjective.

Let $\phi = (\phi^{(r)},\phi^{(r-1)},\ldots,\phi^{(0)}) \in S_{r,Q}$, then
$\zeta = (\zeta^{(r-1)},\zeta^{(r-2)},\ldots,\zeta^{(0)}) \in S_{r-1,Q}$ using (4.6d) (ζ satisfies (i) and (ii) for $2 \leq \rho \leq r-1$ since ϕ does for $2 \leq \rho \leq r$) .

Now, by (4.6d(ii))

$$\beta^r\phi^{(r)} = \phi^{(r-2)}\beta^r$$

and hence $\phi^{(r)}(\ker\beta^r) \subseteq \ker\beta^r$ so that we can define $\theta \in \mathrm{End}_Q(\ker_Q\beta^r)$ by $\theta(y) := \phi^{(r)}(y)$ $(y \in \ker_Q\beta^r)$.

Since the action of $\phi^{(r)}$ commutes with the action of $G(r)$, $\theta \in \mathrm{End}_{QG(r)}(\ker_Q\beta^r)$ and it is now easy to see that $\kappa(\theta,\zeta) = \phi$, proving that κ is surjective.

Now define $\kappa' : S_{r,Q} \to \mathrm{End}_{QG(r)}(\ker_Q \beta^r) \oplus S_{r-1,Q}$ to be the inverse of κ ; the last calculation above shows that $\kappa'(\phi) = (\theta,(\phi^{(r-1)},\phi^{(r-2)},\ldots,\phi^{(0)}))$ ie. the composite of κ' with the projection onto the second factor, $S_{r-1,Q}$ is given by

$$(\phi^{(r)},\phi^{(r-1)},\ldots,\phi^{(0)}) \to (\phi^{(r-1)},\phi^{(r-2)},\ldots,\phi^{(0)}) .$$

This completes the proof.

4.6g Corollary

The Q-algebra $S_{r,Q}$ is semisimple.

Proof

We use induction on r . If $r = 0$, then $S_{0,Q} \cong Q$ is semisimple and if $r = 1$ $S_{1,Q} = S_{1,Q}(G)$ is isomorphic to $\mathrm{End}_Q(E_Q)\oplus Q$, a semi-simple Q-algebra.

Now, assume $S_{r-1,Q}$ is semisimple. We have

$$S_{r,Q} \cong \mathrm{End}_{QG(r)}(\ker_Q \beta^r) \oplus S_{r-1,Q}$$

by (4.6f).

Now $QG(r)$ is semisimple which implies that $\ker_Q \beta^r$ is a completely reducible $QG(r)$ module, but the endomorphism algebra of a completely reducible module is semisimple and it follows that $S_{r,Q}$ is semisimple.

4.6h Corollary to 4.6f

We have $I_{r,Q} \cap Q_{r-1}[G] = I_{r-1,Q}$,

and hence $I_{r,Q} \cap Q_s[G] = I_{s,Q}$ $(s \leq r)$.

Proof

Since the map $S_{r,Q} \to S_{r-1,Q}$ taking $(\phi^{(r)},\phi^{(r-1)},\ldots,\phi^{(0)})$ to $(\phi^{(r-1)},\phi^{(r-2)},\ldots,\phi^{(0)})$ is surjective the map

$$B_{r-1,Q} \to B_{r,Q}$$

$$c_{hk} + I_{r-1,Q} \mapsto c_{hk} + I_{r,Q} \qquad (h,k \in T_r(n))$$

is injective and the corollary follows.

4.6i Theorem

We have $S_{r,Q} = S_{r,Q}(\Gamma)$ and hence $S_{r,Q}(\Gamma)$ is semisimple.

Proof

To prove the theorem we need only show that $J_{r,Q} \subseteq I_{r,Q}$ since then $S_{r,Q} \subseteq S_{r,Q}(\Gamma)$ and we already have the reverse inclusion. Let $x \in J_{r,Q}$, then $x \in I_{t,Q} \cap Q_r[G]$ for some t (2.6e). If $t \leq r$ we are finished, if $t > r$ apply (4.6h) to get $x \in I_{r,Q}$.

§5. Representation theory 1

5.1 Representation theory of G

This section is a summary of parts of §3, §4 and §5 of [G].

Recall that by (0.1i) to study polynomial modules in $M_K(G)$ it is enough to study the homogeneous ones ie. those in $M_K^r(G)$ $(r \geq 0)$. Further, by (0.2f) the categories $M_K^r(G)$ and $\mathrm{mod}(S_K^r(G))$ are equivalent.

We have defined the set of weights of G, $\Lambda(G)$ to consist of all n-tuples of non negative integers (1.1,p.17). The symmetric group $G(n)$ acts on the left of $\Lambda(G)$ by:

5.1a

$$\pi(\lambda_1,\lambda_2,\ldots,\lambda_n) = (\lambda_{\pi^{-1}(1)},\lambda_{\pi^{-1}(2)},\ldots,\lambda_{\pi^{-1}(n)})$$

$$(\pi \in G(n)\ ,\ (\lambda_1,\lambda_2,\ldots,\lambda_n) \in \Lambda(G))\ .$$

Each $G(n)$ orbit of $\Lambda(G)$ contains exactly one dominant weight ie. a weight $(\lambda_1,\lambda_2,\ldots,\lambda_n)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. We denote by $\Lambda^+(G)$ the set of all dominant weights and order them, when necessary, lexicographically. For any integer $r > 0$ define $\Lambda^r(G) \subset \Lambda(G)$ to be the set of weights $(\lambda_1,\lambda_2,\ldots,\lambda_n)$ such that $\lambda_1+\lambda_2+\ldots+\lambda_n = r$ and denote by $\Lambda^{r,+}(G)$ the set $\Lambda^r(G) \cap \Lambda^+(G)$. (Define $\Lambda^{0,+}(G) = \Lambda^0(G)$ to be the set consisting of the empty partition.) Let $i \in I(n,r)$, then we say i is of weight $\lambda \in \Lambda^r(G)$, and write $i \in \lambda$, if for each $\mu \in \underline{n}$, λ_μ is the number of i_ρ $(\rho \in \underline{r})$ equal to μ. Thus, if $n = 4$, $r = 3$ then (1,4,2) and (2,1,4) in $I(4,3)$ are both of weight $(1,1,0,1) \in \Lambda^3(G_4)$.

5.1b

Recall that $S_K^r(G)$ is spanned by elements ζ_{ij} ($i,j \in I(n,r)$) such that for $h,k \in I(n,r)$ $\zeta_{ij} = \zeta_{hk}$ iff $(i,j) \sim (h,k)$ (0.2e). Now, clearly $i,j \in I(n,r)$ are of the same weight iff there is some $\pi \in G(r)$ such that $i\pi = j$ ie. iff $i \sim j$. It follows that in that case $\zeta_{ii} = \zeta_{jj}$. Thus, for $\lambda \in \Lambda^r(G)$ we can denote unambiguously by ζ_λ the element $\zeta_{ii} \in S_K^r(G)$ where $i \in \lambda$.

5.1c

Recall that $K[T_G]$ is a free polynomial algebra on generators $\{c_{\mu\mu}^T : \mu\in\underline{n}\}$ (1.1c). Define $K^r[T_G]$ to be the K-subspace of $K[T_G]$ spanned by homogeneous polynomials of degree r in the $c_{\mu\mu}^T$ $(\mu\in\underline{n})$. Then the surjective morphism of K-coalgebras

$$\Psi_K^r(T) : K^r[G] \to K^r[T_G]$$

given by restriction from G to T_G induces an injective morphism of K-algebras

$$S_K^r(T_G) \to S_K^r(G)$$

▲

We identify $S_K^r(T_G)$ with its image in $S_K^r(G)$.

5.1d Lemma

The subalgebra $S_K^r(T_G)$ of $S_K^r(G)$ has basis $\{\zeta_\lambda : \lambda \in \Lambda^r(G)\}$.

Proof

Clearly each $\zeta_\lambda \in S_K^r(T_G)$ since it annihilates the kernel of $\Psi_K^r(T)$ (which is generated by $\{c_{\mu\nu} : \mu,\nu \in \underline{n},\ \mu \neq \nu\}$) . But the ζ_λ $(\lambda \in \Lambda^r(G))$ are linearly independent and since weights in $\Lambda^r(G)$ are in one to one correspondence with monomials in $K^r[T_G]$ (1.1d) they must span $S_K^r(T_G)$.

5.1e

Let 1_r denote the identity in $S_K^r(G)$. Then

$$1_r = \sum_{\lambda \in \Lambda^r(G)} \zeta_\lambda \qquad \ldots\ldots \qquad (A).$$

Further, if $\lambda, \gamma \in \Lambda^r(G)$

$$\zeta_\lambda \zeta_\gamma = \zeta_\gamma \zeta_\lambda = \begin{cases} \zeta_\lambda & \lambda = \gamma \\ 0 & \lambda \neq \gamma \end{cases} \qquad \ldots\ldots \qquad (B).$$

Together (A) and (B) imply that (A) is an orthogonal idempotent decomposition of the identity in $S_K^r(G)$. Let $V \in M_K^r(G)$, then

$$V = \sum_{\lambda \in \Lambda^r(G)}^{\oplus} \zeta_\lambda V$$

and in fact [G,§3.2]

$$\zeta_\lambda V = \{v \in V : t.v = \chi_\lambda(t).v \quad \forall t \in T_G\}$$

where $\chi_\lambda : T_G \to K$ is the character defined in (1.1e).

Therefore $\zeta_\lambda V$ is a 'weight space' in the usual sense, we denote it in the usual way by V^λ .

<u>5.1f</u> <u>Theorem</u> (Schur, Chevalley: see [G,3.5a])

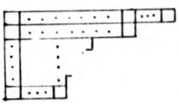
For each $\lambda \in \Lambda^{r,+}(G)$ there exists an absolutely irreducible module $F_{\lambda,K} \in M_K^r(G)$ with λ as its highest weight ie. if $F^\mu_{\lambda,K} \neq 0$ for some $\mu \in \Lambda^{r,+}(G)$ then $\mu \leq \lambda$. Moreover $\dim_K F^\lambda_{\lambda,K} = 1$.

Every irreducible module in $M_K^r(G)$ is isomorphic to precisely one of these.

<u>5.1g</u>

Let $\lambda \in \Lambda^{r,+}(G)$. We define a 'λ shape' $[\lambda]$ to be a diagram consisting of λ_μ boxes in the μ^{th} row ie.

$[\lambda]:=$



λ_1-boxes
λ_2-boxes
$\vdots$
λ_n-boxes

(if $r = 0$ $[\lambda]$ is the 'empty' shape).

Define $T : [\lambda] \to \underline{r}$ to be the map which identifies each box of $[\lambda]$ with an element of $\underline{r}$ such that the first row contains $1,2,\ldots,\lambda_1$ from left to right, the second row $\lambda_1+1,\lambda_1+2,\ldots,\lambda_1+\lambda_2$ from left to right etc. For example, if $\lambda = (3,2,2)$ then we can write

$$T^{\lambda} := T([\lambda]) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2}\end{array}$$

For $i \in I(n,r)$ define the λ-tableaux T_i to be the composition of $T : [\lambda] \to \underline{r}$ with $i : \underline{r} \to \underline{n}$. For example, when $\lambda = (3,2,2)$, $i = (1,3,4,2,4,3,5)$ then

$$T_i = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 4 \\ \cline{1-2} 3 & 5 \\ \cline{1-2}\end{array}$$

We say T_i is standard if entries increase strictly down columns and are non decreasing along rows, for example the tableaux above is standard.

Let $T_{h(\lambda)}$ be the λ-tableaux with μ's in the μ^{th} row (this defines $h(\lambda) \in I(n,r)$) and denote by $R(\lambda)$ the subgroup of $G(r)$ fixing $h(\lambda)$. Similarily, let $T_{k(\lambda)}$ be the λ-tableaux with μ's in the μ^{th} column and denote by $C(\lambda)$ the subgroup of $G(r)$ fixing $k(\lambda) \in I(n,r)$. Clearly $T_{h(\lambda)}$ is standard and $h(\lambda)$ is of weight λ. It is not hard to see that if $i \in \lambda$ with T_i standard then $i = h(\lambda)$.

<u>Example</u>

When $\lambda = (3,2,2)$

$$T_{h(\lambda)} = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 3 \\ \cline{1-2}\end{array} \qquad T_{k(\lambda)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 1 & 2 \\ \cline{1-2} 1 & 2 \\ \cline{1-2}\end{array}$$

5.1h

The modules $D_{\lambda,K}$: Let $\lambda \in \Lambda^{r,+}(G)$, then for $i,j \in I(n,r)$ we define the bideterminant $(T_i : T_j) \in K^r[G]$ to be the sum

$$\sum_{\sigma \in C(\lambda)} \varepsilon(\sigma) c_{i,j\sigma}$$

where $\varepsilon(\sigma)$ is the 'sign' of σ.

Define $D_{\lambda,K}$ to be the K-span of all the $(T_{h(\lambda)} : T_i)$ $(i \in I(n,r))$.

5.1i Theorem (see for example [G,4.5a]

The set $\{(T_{h(\lambda)} : T_i) : T_i$ standard$\}$ is a basis of $D_{\lambda,K}$.

Now $K^r[G]$ can be given a KG module structure by extending to the whole of $K^r[G]$ the action:

$$g.c_{ij} = \sum_{h \in I(n,r)} c_{hj}(g) c_{ih} \quad (g \in G, i,j \in I(n,r))$$

and hence, for $\phi \in S_K^r(G)$:

$$\phi.c_{ij} = \sum_{h \in I(n,r)} \phi(c_{hj}) c_{ih} \quad (i,j \in I(n,r)) .$$

It follows that for $j \in I(n,r)$:

$$\phi.(T_{h(\lambda)} : T_j) = \sum_{i \in I(n,r)} \phi(c_{ij})(T_{h(\lambda)} : T_i) .$$

Thus $D_{\lambda,K}$ is a KG (equivalently $S_K^r(G)$) submodule of $K^r[G]$ and for $\gamma \in \Lambda^r(G)$ then

$$\xi_\gamma D_{\lambda,K} = D^\gamma_{\lambda,K} = K\text{-span}\{(T_{h(\lambda)}:T_i) : i \in \gamma,\ T_i \text{ standard}\} ,$$

giving $D^\lambda_{\lambda,K} = K.(T_{h(\lambda)} : T_{h(\lambda)})$.

5.1j The Carter-Lusztig 'Weyl' module $V_{\lambda,K}$:

For any subset H of $G(r)$ we denote by $\{H\}$ the sum $\sum_{\sigma \in H} \varepsilon(\sigma)\sigma$ in $KG(r)$.

For $\lambda \in \Lambda^{r,+}(G)$ we define the Carter-Lusztig Weyl module $V_{\lambda,K}$ to be the KG-submodule of E_K^r generated by $f_{h(\lambda)} := e_{h(\lambda)}\{C(\lambda)\}$. [CL,p.211, 222].

5.1k Theorem [CL, theorem 3.5]

The KG-module $V_{\lambda,K}$ has K-basis

$$\{b_i := \xi_{i,h(\lambda)}.f_{h(\lambda)} : T_i \text{ standard}\} .$$

As a consequence we have

(i) Let $\gamma \in \Lambda^r(G)$, then

$$\zeta_\gamma V_{\lambda,K} = V^\gamma_{\lambda,K} = K\text{-span}\,\{b_i : i\in\gamma,\ T_i \text{ standard}\}$$

and hence $V^\lambda_{\lambda,K} = K.f_{h(\lambda)}$.

(ii) $V_{\lambda,K}$ has a unique maximal proper submodule $M_{\lambda,K}$. Clearly $f_{h(\lambda)} \notin M_{\lambda,K}$ and $V_{\lambda,K}/M_{\lambda,K}$ is therefore generated by a highest weight vector $f_{h(\lambda)} + M_{\lambda,K}$ of weight λ. Hence $F_{\lambda,K} \cong V_{\lambda,K}/M_{\lambda,K}$.

(iii) When char $K = 0$, $M_{\lambda,K} = (0)$, hence $V_{\lambda,K} \cong F_{\lambda,K}$.

<u>Duality between $D_{\lambda,K}$ and $V_{\lambda,K}$</u>.

Let $\theta_{\lambda,K} : E^r_K \twoheadrightarrow D_{\lambda,K}$ be the epimorphism of KG-modules given by

<u>5.1ℓ</u>

$$\theta_{\lambda,K}(e_i) = (T_{h(\lambda)} : T_i) \qquad (i \in I(n,r))$$

and denote by $N_{\lambda,K}$ the kernel of $\theta_{\lambda,K}$.

<u>5.1m</u> <u>Proposition</u> [G,5.2a]

The KG-submodule $N_{\lambda,K}$ of E^r_K is spanned by the subset $R_1 \cup R_2 \cup R_3$ of E^r_K, where

(i) R_1 consists of all e_i $(i \in I(n,r))$ such that T_i has a repeated entry in some column.

(ii) R_2 consists of all $e_i - \varepsilon(\sigma)e_{i\sigma}$ $(i \in I(n,r)\ ,\ \sigma \in C(\lambda))$.

(iii) (Garnir relations) R_3 consists of all the elements $e_i(G(J))$ $(i \in I(n,r))$ where $G(J)$ is a transversal of a set of cosets in $G(r)$ dependent on J a subset of the $(h+1)^{th}$ column of T^λ $(h \in \underline{r-1})$ (for exact definition, which is not needed here, see for example [G,4.6a]).

Following (4.5,p.77) we define $A : E_K \times E_K \to K$ to be the non singular symmetric bilinear form given by

$$A(e_\mu, e_\nu) = \delta_{\mu\nu} . 1_K \qquad (\mu,\nu \in \underline{n}) \ .$$

This form induces a non singular symmetric bilinear form $A^r : E_K^r \times E_K^r \to K$ as usual by defining for $i,j \in I(n,r)$:

$$A^r(e_i, e_j) = \prod_{\rho \in \underline{r}} A(e_{i_\rho}, e_{j_\rho}) \ .$$

This form is 'contravariant' with respect to the K algebra antiautomorphism $w^* : S_K^r(G) \to S_K^r(G)$ given by $w^*(\zeta_{ij}) = \zeta_{ji}$ $(i,j \in I(n,r))$ ie.

$$A^r(\phi(x), y) = A^r(x, w^*\phi(y))$$

for all $x,y \in E_K^r$, $\phi \in S_K^r(G)$.

5.1n Theorem [G,5.2]

Let $\lambda \in \Lambda^{r,+}(G)$. The KG-module $V_{\lambda,K}$ is the orthogonal complement of $N_{\lambda,K}$ with respect to the form A^r on E_K^r . It follows that we may define a non-singular bilinear form

$$(\; , \;) : V_{\lambda,K} \times D_{\lambda,K} \to K$$

by $(x,\theta_{\lambda,K}(y)) = A^r(x,y) \quad (x \in V_{\lambda,K}, y \in E_K^r)$. (See (5.1ℓ) for the definition of $\theta_{\lambda,K}$) .

This form is w^*-contravariant, ie. for $\phi \in S_K^r(G)$, $x \in V_{\lambda,K}$, $y \in D_{\lambda,K}$:

$$(\phi(x),y) = (x,w^*\phi(y)) \quad .$$

Now, since by (5.1n) $D_{\lambda,K}$ is dual to $V_{\lambda,K}$ we have the following corollary to (5.1k):

<u>5.1o</u>

$D_{\lambda,K}$ has a unique minimal submodule isomorphic to $F_{\lambda,K}$, (this uses the important fact that the form (,) is w^*-contravariant), [G,5.4c]. In particular, if char K = 0 then $D_{\lambda,K} \cong F_{\lambda,K}$ as KG-modules.

<u>5.1p</u> <u>Remark</u>

The QG module $D_{\lambda,Q}$ $(\lambda \in \Lambda^{r,+}(G))$ has 'admissible Z-form'

$$D_{\lambda,Z} := \Sigma Z(T_{h(\lambda)}:T_i)$$

where the sum is over all $i \in I(n,r)$ with T_i standard.

(We define admissible Z-form for QG modules in the same way as for $Q\Gamma$-modules (3.2).) Further, $D_{\lambda,Z} \otimes_Z K$ is isomorphic as a KG module to $D_{\lambda,K}$ using the isomorphism (3.1d) : $Z_+[G_Q] \otimes K \xrightarrow{\sim} K_+[G_K]$.

Also, it can be shown [CL,Theorem 3.5], [G,5.4e] that $V_{\lambda,Q}$ has an admissible Z-form

$$V_{\lambda,Z} := Z\text{-span } \{b_i : i \in I(n,r),\ T_i \text{ standard}\}$$

and $V_{\lambda,Z} \otimes_Z K$ is isomorphic as KG-module to $V_{\lambda,K}$.

Now, for $i,j \in I(n,r)$, T_i, T_j standard it is easy to see that

$$(b_i, (T_{h(\lambda)}:T_j)) = \Omega_{ij} \in Z \quad .$$

In fact the matrix (Ω_{ij}) is unimodular [D,p.74] so that the form

$$(\ ,\ ) : V_{\lambda,Z} \times D_{\lambda,Z} \to Z$$

induced by restriction of $(\ ,\ ) : V_{\lambda,Q} \times D_{\lambda,Q} \to Q$ is non singular.

It is the main aim of (§6) to find a Z-form of the Γ versions of the $V_{\lambda,Q}$ and $D_{\lambda,Q}$ which are described below.

As an aside, we prove the following two results which will be required later:

5.1q Lemma

For any infinite field K, $r > 0$, $i,j \in I(n,r)$

$$A^r(e_i,e_j) = B^r(e_i,J^r(e_j))$$

where $B: E_K \times E_K \to K$ is the bilinear form which gives rise to Γ and $J \in \Gamma$ is the matrix of B.

Proof

Clearly $A(e_\mu,e_\nu) = \delta_{\mu\nu}.1_K = B(e_\mu,J(e_\nu))$. The general case follows.

5.1r Corollary

The KG-module $V_{\lambda,K}$ is the orthogonal complement of $N_{\lambda,K}$ with respect to the form $B^r: E_K^r \times E_K^r \to K$.

Proof

Suppose $A^r(x,N_{\lambda,K}) = 0$ then $B^r(x,J(N_{\lambda,K})) = 0$, but $J(N_{\lambda,K}) = N_{\lambda,K}$ since $J \in G$. Hence $B^r(x,N_{\lambda,K}) = 0$. The converse is similar.

5.2 Weights of $\Gamma = O_{2\ell+1}$

Let K be any infinite field, $n = 2\ell+1$.

We defined in (1.3,p.32) the set $\Lambda(\Gamma)$ of weights to be the set of all $\ell+1$-tuples $(\alpha_1,\alpha_2,\ldots,\alpha_{\ell+1})$ with $\alpha_\rho \in Z$ $(\rho \in \underline{\ell})$ and $\alpha_{\ell+1} \in \{0,1\}$.

We say $\alpha \in \Lambda(\Gamma)$ is dominant if $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_\ell \geq 0$.

5.2a

Define a map $\Lambda(G) \twoheadrightarrow \Lambda(\Gamma)$ taking $\lambda \in \Lambda(G)$ to $\lambda^* = (\lambda_1-\lambda_n, \lambda_2-\lambda_{n-1}, \ldots, \lambda_\ell-\lambda_{\ell+2}, \bar{\lambda}_{\ell+1}) \in \Lambda(\Gamma)$, where $\bar{\lambda}_{\ell+1}$ is $\lambda_{\ell+1}$ modulo 2. For a positive integer r define $\Lambda_r(\Gamma)$ to be the image of $\Lambda_r(G) := \bigcup_{0\leq s\leq r} \Lambda^s(G)$ under this map ($\Lambda_0(\Gamma)$ has only one element, viz the empty partition). Also, define $\Lambda_r^+(\Gamma)$ to be the set of dominant weights in $\Lambda_r(\Gamma)$, then it is clear that $\Lambda_r^+(\Gamma)$ is the image of $\Lambda^{0,+}(G) \cup \ldots \cup \Lambda^{r,+}(G)$ under the map $\lambda \to \lambda^*$.

Recall that $K[T_\Gamma]$ has basis consisting of characters χ_α^Γ $(\alpha\in\Lambda(\Gamma))$ (1.3,p.32). It is easy to see that for $\lambda\in\Lambda(G)$ the restriction of the character χ_λ of T_G to T_Γ is precisely $\chi_{\lambda^*}^\Gamma$.

We define $K_r[T_\Gamma]$ $(r\geq 0)$ to be the image of $K_r[G]$ under the restriction map $K_+[G] \to K[T_\Gamma]$ (0.3,p.11). Then $K_r[T_\Gamma]$ has a basis of characters $\{\chi_\alpha^\Gamma : \alpha\in\Lambda_r(\Gamma)\}$. The restriction map

$$K_r[G] \twoheadrightarrow K_r[T_\Gamma]$$

is clearly surjective and therefore induces an injective morphism of K-algebras

$$S_{r,K}(T_\Gamma) \to S_{r,K}(G).$$

We identify $S_{r,K}(T_\Gamma)$ with its image in $S_{r,K}(G)$. Then $S_{r,K}(T_\Gamma) \subset S_{r,K}(T_G)$ since $T_\Gamma \subset T_G$.

5.2b Proposition

The subalgebra $S_{r,K}(T_\Gamma)$ of $S_{r,K}(G)$ has basis $\{\phi_\alpha : \alpha\epsilon\Lambda_r(\Gamma)\}$ where for each $\alpha\epsilon\Lambda_r(\Gamma)$

$$\phi_\alpha = \sum_{\substack{\lambda\epsilon\Lambda_r(G) \\ \lambda^* = \alpha}} \zeta_\lambda .$$

Proof

Recall that we have defined the map $\epsilon_r : KG \to S_{r,K}(G)$ (0.2,p.10). If $g\epsilon G$ then $\epsilon_r(g) : K_r[G] \to K$ is evaluation at g . By (0.3b) $\epsilon_r(KT_G) = S_{r,K}(T_G)$ and $\epsilon_r(KT_\Gamma) = S_{r,K}(T_\Gamma)$. Now, if $t = \mathrm{diag}(t_1,t_2,\dots,t_n) \in T_G$ then by (5.1d)

$$\epsilon_r(t) = \sum_{\lambda\epsilon\Lambda_r(G)} a_\lambda \zeta_\lambda \qquad (a_\lambda \epsilon K) \qquad \dots\dots \qquad (A)$$

If $i\epsilon I(n,r)$ and $i \in \gamma \in \Lambda_r(G)$ then

$$\epsilon_r(t)(c_{ii}) = t_1^{\gamma_1} t_2^{\gamma_2} \dots t_\Gamma^{\gamma_\Gamma} = \chi_\gamma(t)$$

and also

$$(\Sigma a_\lambda \zeta_\lambda)(c_{ii}) = a_\gamma \zeta_\gamma(c_{ii}) = a_\gamma$$

so that $a_\gamma = \chi_\gamma(t)$.

Now suppose $t \in T_\Gamma$, then if $\lambda,\gamma \in \Lambda_r(G)$ are such that $\lambda^* = \gamma^* = \alpha \in \Lambda_r(\Gamma)$ we have $\chi_\lambda(t) = \chi_\gamma(t) = \chi_\alpha^\Gamma(t)$. Thus (A) can be rewritten as:

$$\epsilon_r(t) = \sum_{\alpha\in\Lambda_r(\Gamma)} \chi_\alpha^\Gamma(t)\phi_\alpha \qquad (t \in T_\Gamma)$$

where $\phi_\alpha = \sum_{\substack{\lambda\in\Lambda_r(G)\\ \lambda^*=\alpha}} \zeta_\chi$.

Now, $\{\chi_\alpha^\Gamma : \alpha\in\Lambda_r(\Gamma)\}$ are linearly independent in K^{T_Γ} and it follows that each ϕ_α is a linear combination of $\epsilon_r(t)$'s $(t\in T_\Gamma)$ and hence $\phi_\alpha \in S_{r,K}(T_\Gamma)$. We are now finished, since the ϕ_α $(\alpha\in\Lambda_r(\Gamma))$ are linearly independent in K^G and the number of them equals the dimension of $K_r[T_\Gamma]$ ie. the dimension of $S_{r,K}(T_\Gamma)$.

5.2c Lemma

Suppose $\lambda\in\Lambda^r(G)$ with $\lambda_{\ell+1} = \lambda_{\ell+2} = \ldots = \lambda_n = 0$ then $\phi_{\lambda^*} = \zeta_\lambda$.

Proof

We have $\lambda^* = (\lambda_1,\lambda_2,\ldots,\lambda_\ell,0)$. Suppose $\gamma\in\Lambda_r(G)$ with $\gamma^* = \lambda^*$ then $\gamma_\rho - \gamma_{\bar\rho} = \lambda_\rho$ $(\rho\in\underline{\ell})$ and $\gamma_{\ell+1} \equiv 0$ modulo 2. This implies that $\sum_{\mu\in\underline{n}} \gamma_\mu \geq \lambda_1 + \lambda_2 + \ldots + \lambda_\ell = r$ with equality iff $\gamma = \lambda$. This proves the lemma.

Let 1_r denote the identity of $S_{r,K}(G)$, then (5.1e) says $1_r = \sum_{\lambda\in\Lambda_r(G)} \zeta_\lambda$ is an orthogonal idempotent decomposition of 1_r . By gathering together terms we can write

5.2d

$$1_r = \sum_{\alpha\in\Lambda_r(\Gamma)} \phi_\alpha .$$

Clearly $\phi_\alpha\phi_\beta = \phi_\beta\phi_\alpha$ is zero unless $\alpha = \beta$ when it is equal to ϕ_α . Thus (5.2d) is an orthogonal idempotent decomposition in $S_{r,K}(T_\Gamma) \subset S_{r,K}(G)$.

Let $V \in M_{r,K}(\Gamma)$, then

$$V = \sum_{\alpha\in\Lambda_r(\Gamma)}^{\oplus} \phi_\alpha V$$

$V^\alpha := \phi_\alpha V = \{v\in V \mid t.v = \chi_\alpha^\Gamma(t).v \text{ for all } t\in T_\Gamma\}$ using (5.1e), (5.2d).

5.3 Irreducible modules of $S_{r,Q}(\Gamma)$

In this section we find the irreducible modules of $S_{r,Q}(\Gamma)$ as submodules of E_Q^ρ $(0 \leq \rho \leq r)$. This was first achieved by Weyl [W,5.7G]. Recall that by (4.6f,i) we have:

$$S_{r,Q}(\Gamma) \cong \sum_{\rho=2}^{r\,\oplus} \mathrm{End}_{QG(\rho)}(\ker_Q\beta^\rho) \oplus S_{1,Q}(\Gamma)$$

with $S_{1,Q}(\Gamma) = S_{1,Q}(G) = S_Q^1(G) \oplus S_Q^0(G)$.

Hence, since the irreducible QG-modules of degree 1 and 0 remain irreducible on restriction to $Q\Gamma$, it is enough to find the irreducibles of the (semisimple) Q-algebras

$$A_\rho := \mathrm{End}_{QG(\rho)}(\ker_Q \beta^\rho) \qquad (2 \leq \rho \leq r) \quad .$$

Since $r \geq 2$ is arbitrary we can work with A_r .

Let $h_r \in \mathrm{End}_Q(E_Q^r)$ $(r \geq 2)$ be the unique idempotent such that $h_r(E_Q^r) = \ker_Q \beta^r$, $h_r(\mathrm{Im}_Q \beta^{r,+}) = 0$ (4.5f). Since $\ker_Q \beta^r$ and $\mathrm{Im}_Q \beta^{r,+}$ are $QG(r)$ submodules of E_Q^r (4.4c,e), $h_r \in \mathrm{End}_{QG(r)}(E_Q^r)$, and it follows that:

5.3a

$$h_r \mathrm{End}_{QG(r)}(E_Q^r) h_r \cong \mathrm{End}_{QG(r)}(\ker_Q \beta^r) \; .$$

Under this isomorphism A_r is mapped to the set of elements of $\mathrm{End}_{QG(r)}(E_Q^r)$ which are zero on $\mathrm{Im}_Q \beta^{r,+}$.

Now $S_Q^r(G)$ is naturally isomorphic to $\mathrm{End}_{QG(r)}(E_Q^r)$ (4.2a) and if we identify h_r with its image in $S_Q^r(G)$ then

$$h_r S_Q^r(G) h_r \cong A_r \quad .$$

Identify A_r with this subalgebra of $S_Q^r(G)$.

Let $V \in M_Q^r(G)$, then $h_r V$ is an A_r-module and we have :

5.3b Theorem

Let $\{V_{\lambda,Q} : \lambda \in \Lambda^{r,+}(G)\}$ be a full set of irreducible $S^r_Q(G)$-modules (the Carter-Lusztig modules). Then

(i) for each $\lambda \epsilon \Lambda^{r,+}(G)$, $h_r V_{\lambda,Q}$ is either zero or irreducible,

(ii) for $\lambda \neq \gamma \in \Lambda^{r,+}(G)$ with $h_r V_{\lambda,Q} \neq 0 \neq h_r V_{\gamma,Q}$ then $h_r V_{\lambda,Q} \ncong h_r V_{\gamma,Q}$ as A_r modules,

(iii) let $\Lambda = \{\lambda \epsilon \Lambda^{r,+}(G) : h_r V_{\lambda,Q} \neq 0\}$ then $\{h_r V_{\lambda,Q}\}_{\lambda \epsilon \Lambda}$ is a full set of irreducible A_r-modules.

Proof This is an application of 'eSe' theory as developed by T. Martins [M].

It remains to show for which $\lambda \epsilon \Lambda^{r,+}(G)$, $h_r V_{\lambda,Q} \neq 0$.

Let $\lambda \epsilon \Lambda^{r,+}(G)$, we say λ is 'permissible' if the sum of the lengths of the first two columns of the λ-shape $[\lambda]$ is $\leq n$.

5.3c Theorem

The A_r-module $h_r V_{\lambda,Q} \neq 0$ iff $\lambda \epsilon \Lambda^{r,+}(G)$ is permissible.

It follows that

$$\{h_\rho V_{\lambda,Q} : \lambda \epsilon \Lambda^{\rho,+}(G) \text{ permissible, } 0 \leq \rho \leq r\}$$

is a full set of non isomorphic irreducible $S_{r,Q}(r)$ modules, (where we define h_0, h_1 to be the identity maps on E_Q^0, E_Q^1 respectively).

We shall prove the theorem using two lemmas.

5.3d Lemma Weyl [W,5.7B]

Let K be any infinite field $r \geq 2$, $\lambda \varepsilon \Lambda^{r,+}(G)$. Then $E_K^r(C(\lambda)) \subseteq \mathrm{Im}_K \beta^{r,+}$ if λ is not permissible.

5.3e Remark

Weyl proves this lemma using the form $A : E_Q \times E_Q \to Q$ (4.5,p.77). At first sight his proof seems to rely on invariant theory but in fact it not only does not use invariant theory but the proof works for any infinite field K. It is therefore perhaps worthwhile to give an expanded version of Weyl's proof adapted to suit our needs.

Proof (of (5.3d))

Let $\lambda \varepsilon \Lambda^{r,+}(G)$ and suppose that the lengths of the columns of the λ shape $[\lambda]$ are $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r$. If λ is not permissible, $\mu_1 + \mu_2 > n$. Since it is clear from the definition of $C(\lambda)$ (5.1g) that $C(\lambda) \cong G(\mu_1) \times G(\mu_2) \times \ldots \times G(\mu_r)$ ($G(\mu_\rho)$ acting on column ρ of T^λ of length μ_ρ ($\rho \varepsilon \underline{r}$)) we can identify $E_K^r(C(\lambda))$ with the tensor product

$$E_K^{\mu_1}(G(\mu_1)) \otimes E_K^{\mu_2}(G(\mu_2)) \otimes \ldots \otimes E_K^{\mu_r}(G(\mu_r)) \subset E_K^r .$$

(Note that if $\mu_\rho = 0$ we define $G(0)$ to be empty.)

Consider first $E_K^{\mu_1}(G(\mu_1)) \otimes E_K^{\mu_2}(G(\mu_2)) \subseteq E_K^{\mu_1 + \mu_2}$.

If we show that each $x \in E_K^{\mu_1}(G(\mu_1)) \otimes E_K^{\mu_2}(G(\mu_2))$ is a sum of terms, each in the image of some

$$\beta_{ab}^{\mu_1+\mu_2,+} : E_K^{\mu_1+\mu_2-2} \to E_K^{\mu_1+\mu_2} \qquad 1 \leq a < b \leq \mu_1 + \mu_2$$

it will follow that each $y \in E_K^r(C(\lambda))$ is in $\mathrm{Im}_K\beta^{r,+}$ since

$$\mathrm{Im}_K\beta^{r,+} = \sum_{1\leq a<b\leq r} K.(\mathrm{Im}\beta_{ab}^{r,+} : E_K^{r-2} \to E_K^r)$$

and $y = x \otimes z$ for some $x \in E_K^{\mu_1}(G(\mu_1)) \otimes E_K^{\mu_2}(G(\mu_2))$
$z \in E_K^{\mu_3}(G(\mu_3)) \otimes \ldots \otimes E_K^{\mu_r}(G(\mu_r))$.

Hence, to prove the lemma it is enough to work with $E_K^{\mu_1}(G(\mu_1)) \otimes E_K^{\mu_2}(G(\mu_2))$.

We can identify $E_K^{\mu_1}$, $E_K^{\mu_2}$ and $E_K^{\mu_1+\mu_2}$ with subspaces of the free polynomial ring $K[X,Y] := K[X_\gamma^\alpha, Y_\gamma^\beta : \alpha\in\underline{\mu}_1, \beta\in\underline{\mu}_2, \gamma\in\underline{n}]$ (X_γ^α , Y_γ^β indeterminates over K) as follows:

Extend linearly to the whole of $E_K^{\mu_1}$ the map

$$e_i = e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_{\mu_1}} \to \prod_{\alpha\in\underline{\mu}_1} X_{i_\alpha}^\alpha \qquad (i \in I(n,\mu_1))$$

to the whole of $E_K^{\mu_2}$ the map

$$e_j = e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_{\mu_2}} \to \prod_{\beta\in\underline{\mu}_2} Y_{j_\beta}^\beta \qquad (j \in I(n,\mu_2))$$

and to the whole of $E_K^{\mu_1} \otimes E_K^{\mu_2}$ the map

$$e_i \otimes e_j \to \prod_{\alpha \in \underline{\mu}_1} X^{\alpha}_{i_\alpha} \prod_{\beta \in \underline{\mu}_2} Y^{\beta}_{j_\beta}$$

We identify $E_K^{\mu_1}$, $E_K^{\mu_2}$ and $E_K^{\mu_1+\mu_2}$ with their images in $K[X,Y]$.

Consider $e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_{\mu_1}} \cdot (G(\mu_1)) \in E_K^{\mu_1}$, this is zero unless $i_1, i_2, \ldots, i_{\mu_1}$ are all distinct. So now assume they are all distinct, then under this identification it is the determinant of the $n \times n$ matrix

$$\phi'_X = \left[\begin{array}{cccc:cccc}
X^1_{i_1} & X^1_{i_2} & \cdots & X^1_{i_{\mu_1}} & X^1_{h_1} & \cdots & & X^1_{h_{\nu_1}} \\
\cdot & \cdot & & \cdot & \cdot & & & \cdot \\
\cdot & \cdot & & \cdot & \cdot & & & \cdot \\
\cdot & \cdot & & \cdot & \cdot & & & \cdot \\
X^{\mu_1}_{i_1} & X^{\mu_1}_{i_2} & \cdots & X^{\mu_1}_{i_{\mu_1}} & X^{\mu_1}_{h_1} & \cdots & & X^{\mu_1}_{h_{\nu_1}} \\
\hdashline
 & & & & 1 & 0 & \cdots & 0 \\
 & 0 & & & 0 & 1 & & \\
 & & & & & & \ddots & \\
 & & & & 0 & & & 1
\end{array}\right]$$

where $\{h_1, h_2, \ldots, h_{\nu_1}\} = \underline{n} \setminus \{i_1, i_2, \ldots, i_{\mu_1}\}$, $\mu_1 + \nu_1 = n$, and hence

$\pm$ the determinant of the matrix

$$\Phi_X = \left[\begin{array}{ccccccc} X_1^1 & X_2^1 & . & . & . & . & X_n^1 \\ X_1^2 & X_2^2 & . & . & . & . & X_n^2 \\ \vdots & \vdots & & & & & \vdots \\ X_1^{\mu_1} & X_2^{\mu_1} & . & . & . & . & X_n^{\mu_1} \\ \hline 0 & . & . & . & 0\ 1\ 0 & . & . . . 0 \\ 0\ 0\ 1\ 0 & . & . & . & . & . & 0 \\ 0 & . & . & . & . & . & . . 0\ 1\ 0 . . 0 \end{array}\right]$$

obtained from Φ_X' by permuting the columns (thus the last $n-\mu_1$ rows have all entries zero except for one entry equal to 1 and no column of Φ_X contains more than one entry equal to 1).

Similarily, $e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_{\mu_2}} \cdot (G(\mu_2)) \in E_K^{\mu_2}$, if not zero, is $\pm$ the determinant of the $n \times n$ matrix

$$\Phi_Y = \left[\begin{array}{cccc|cccc} Y_n^1 & Y_n^2 & \ldots & Y_n^{\mu_2} & 0 & . & . & 0 \\ Y_{n-1}^1 & Y_{n-1}^2 & \ldots & Y_{n-1}^{\mu_2} & \vdots & & & \vdots \\ \vdots & \vdots & & \vdots & 0 & . & . & 1 \\ \vdots & \vdots & & \vdots & 1 & . & . & 0 \\ \vdots & \vdots & & \vdots & 0 & . & . & 0 \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ Y_1^1 & Y_1^2 & \ldots & Y_1^{\mu_2} & 0 & . & . & 0 \end{array}\right]$$

the last $n-\mu_2$ columns having exactly one entry equal to 1, the rest zero and no row having more than one entry equal to 1.

It follows that $\det\Phi_X \det\Phi_Y = \det\Phi_X\Phi_Y$ is $\pm$ the image of $e_i(G(\mu_1)) \otimes e_j(G(\mu_2))$ in $K[X,Y]$. Let $\Phi_X\Phi_Y = (a_{\alpha\beta})_{\alpha,\beta\in\underline{n}}$, then

$$\det\Phi_X\Phi_Y = \sum_{\sigma\in G(n)} \epsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \quad .$$

Since $\mu_1+\mu_2 > n$ each product $a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$ contains at least one $a_{\alpha\sigma(\alpha)}$ with $\alpha \leq \mu_1$, $\sigma(\alpha) \leq \mu_2$, hence

$$a_{\alpha\sigma(\alpha)} = \sum_{\nu\in\underline{n}} X_\nu^{\alpha} Y_\nu^{\underline{\sigma}(\alpha)} \qquad \cdot \; \cdot \; \cdot \; \cdot \; \cdot \qquad (A)$$

Now, each $a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$ is homogeneous of total degree $\leq\mu_1+\mu_2$ in the X's and Y's (maximum degree of X's is μ_1, of Y's is μ_2). If it is degree exactly $\mu_1+\mu_2$ it must be the image of some $x \in E_K^{\mu_1+\mu_2}$. But then by (A) this term is divisible by $\sum_{\nu\in\underline{n}} X_\nu^{\alpha} Y_\nu^{\underline{\sigma}(\alpha)}$ (some $\alpha\leq\mu_1, \sigma(\alpha)\leq\mu_2$), so (translating back to $E_K^{\mu_1+\mu_2}$ and using (4.3h)) x is in $\mathrm{Im}\beta_{\alpha\sigma(\alpha)}^{\mu_1+\mu_2,+}$. The remaining terms in the expansion of $\det\Phi_X\Phi_Y$ are all homogeneous of total degree $<\mu_1+\mu_2$ in the X's and Y's and must therefore cancel since $\det\Phi_X\Phi_Y$ is $\pm$ the image of $e_i(G(\mu_1))\otimes e_j(G(\mu_2))$ and therefore homogeneous of total degree $\mu_1+\mu_2$ in the X's and Y's.

This completes the proof.

5.3f Lemma

Let K be any infinite field. If $\lambda \in \Lambda^{r,+}(G)$ $(r \geq 2)$ is permissible then $V_{\lambda,K} \cap \ker_K \beta^r \neq \{0\}$.

Proof

We show that the generator $f_{h(\lambda)} = e_{h(\lambda)}\{C(\lambda)\}$ of $V_{\lambda,K}$ lies in $\ker_K \beta^r$.

If $[\lambda]$ has first column of length $\leq \ell$ then it is clear that $e_{h(\lambda)} \in \ker_K \beta^r$ (recall the definition of $h(\lambda)$ (5.1g)). Hence, for any $(a,b) \in J_o(r)$ by (4.3f) $\beta_{ab}(e_{h(\lambda)}) = B(e_{h(\lambda)_a}, e_{h(\lambda)_b}) e_{h(\lambda)[a,b]} = 0$ because $B(e_\nu, e_{\nu'}) = 0$ for any $\nu, \nu' \leq \ell$. Therefore $f_{h(\lambda)} \in \ker_K \beta^r$ since $\ker_K \beta^r$ is a $KG(r)$ submodule of the $KG(r)$ module E_K^r.

Suppose now that the length of the first column is $m \geq \ell + 1$. Then the second column (and all other columns) is of length at most $n-m \leq \ell$. Thus, if there is some $(a,b) \in J(r)$ such that $\beta_{ab}^r(e_{h(\lambda)}) \neq 0$ then a and b must be elements of the first column of T^λ. For such a λ suppose that $f_{h(\lambda)} \notin \ker_K \beta^r$, then $\beta_{ab}^r(f_{h(\lambda)}) \neq 0$ for some $(a,b) \in J(r)$. Now, the transposition $\tau = (a,b)$ lies in $C(\lambda)$ so that $f_{h(\lambda)} = e_{h(\lambda)}\{C(\lambda)\} = x(1-\tau)$ for some $x \in E_K^r$ (group together terms $e_{h(\lambda)}(\pi - \pi\tau)$). But it can be seen from the definition of β_{ab} that $\beta_{ab}(x) = \beta_{ab}(x\tau)$ for any $x \in E_K^r$, (since $B: E_K \times E_K \to K$ is symmetric). So $\beta_{ab}(f_{h(\lambda)}) = \beta_{ab}(x) - \beta_{ab}(x\tau) = 0$, a contradiction. Hence $f_{h(\lambda)} \in \ker_K \beta^r$, as required.

Proof of Theorem 5.3c

First, notice that for $\lambda \in \Lambda^{\rho,+}(G)$ $(\rho \geq 2)$ then $h_\rho V_{\lambda,Q} = V_{\lambda,Q} \cap \ker_Q \beta^\rho$. Thus (5.3f) implies that if λ is permissible then $h_\rho V_{\lambda,Q} \neq \{0\}$. The converse is true using (5.3d) since $V_{\lambda,Q} \subset E_Q^\rho(C(\lambda))$ by (5.1k). For $\rho = 0,1$, $h_\rho V_{\lambda,Q} = V_{\lambda,Q}$ and all $\lambda \in \Lambda^{0,+}(G) \cup \Lambda^{1,+}(G)$ are permissible.

5.3g

For $\lambda \in \Lambda^{r,+}(G)$ permissible, we denote the irreducible $S_{r,Q}(\Gamma)$ module $h_r V_{\lambda,Q}$ by $V_{\lambda,Q}^\Gamma$.

§6. Representations 2

Throughout K will denote an infinite field, char $K \neq 2$, unless stated otherwise and $n = 2\ell+1$ $(\ell \geq 0)$ when used in connection with $\Gamma_K = O_{2\ell+1}(K)$.

6.1 The irreducible $K\Gamma$ modules $\Lambda^r E_K$

For any integer $r > 0$ we denote the KG module $E_K^r(G(r))$ by $\Lambda^r E_K$ and for $i \in I(n,r)$ the element $e_i(G(r))$ by $\wedge e_i$ or $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r}$.
Then $e_i = \varepsilon(\sigma)\wedge e_{i\sigma}$ for all $\sigma \in G(r)$, so that $\wedge e_i = 0$ if $i_\rho = i_{\rho'}$ for some $\rho \neq \rho' \in \underline{r}$. Thus $\Lambda^r E_K = (0)$ if $r > n$ and for $r \leq n$ $\Lambda^r E_K$ has basis $\{\wedge e_i : 1 \leq i_1 < i_2 < \ldots < i_r \leq n\}$. We also define $\Lambda^0 E_K := K$.

Now, by (5.1k) $V_{1^r,K} = \Lambda^r E_K$ $(r \leq n)$ where $1^r := (1,1,\ldots,1,0,0,\ldots,0) \in \Lambda^{r,+}(G)$ (r 1's).

For $g \in G$, $j \in I(n,r)$ we have

6.1a

$$g.\wedge e_j = \sum_{i \in I(n,r)} c_{ij}(g)\wedge e_i$$

but since $\wedge e_i = 0$ if $i_\rho = i_{\rho'}$, some $\rho \neq \rho' \in \underline{r}$ we can rewrite this sum as

6.1b

$$g.\wedge e_j = \sum_{i \in \bar{I}(n,r)} \Big(\sum_{\sigma \in G(r)} \varepsilon(\sigma) c_{i\sigma,j}(g).\wedge e_i \Big)$$

$$= \sum_{i \in \bar{I}(n,r)} |g|_{ij} \wedge e_i$$

where $\hat{I}(n,r) := \{i \in I(n,r) : 1 \leq i_1 < i_2 < \ldots < i_r \leq n\}$ and $|g|_{ij}$ is the determinant of the matrix

$$\begin{bmatrix} g_{i_1 j_1} & g_{i_1 j_2} & \cdots & g_{i_1 j_r} \\ g_{i_2 j_1} & g_{i_2 j_2} & \cdots & g_{i_2 j_r} \\ \vdots & \vdots & & \vdots \\ g_{i_r j_1} & g_{i_r j_2} & \cdots & g_{i_r j_r} \end{bmatrix}$$

6.1c Theorem

For any integer $0 \leq r \leq n$ and infinite field K, $\Lambda^r E_K$ is irreducible as a KG module and therefore $D_{1^r,K} \cong V_{1^r,K}$.

Proof See for example [G,p.67].

We shall show the corresponding result for $\Gamma = O_{2\ell+1}(K)$ holds also, ie. when considered as a $K\Gamma$-module each $\Lambda^r E_K$ $(0 \leq r \leq n)$ is irreducible (char $K \neq 2$). We first prove the following standard result.

6.1d Lemma

The $K\Gamma$ modules $\Lambda^r E_K$ and $\Lambda^{n-r} E_K \otimes \Lambda^n E_K$ are isomorphic.

Proof

We define a bilinear form

(i) $$F : \Lambda^r E_K \times \Lambda^{n-r} E_K \to K$$

by

(ii) $\quad x_1 \wedge x_2 \wedge . \wedge x_r \wedge J(y_1) \wedge J(y_2) \wedge .. \wedge J(y_{n-r}) = F(x_1 \wedge \ldots \wedge x_r, y_1 \wedge \ldots \wedge y_r) \wedge e_u$

where $x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_{n-r} \in E_K$, $u = (1,2,\ldots,n)$ and J is the matrix of the bilinear form $B : E_K \times E_K \to K$ which defines Γ (0.3c(iii)) so that $J(e_\mu) = e_{\bar{\mu}}$ $(\mu \epsilon \underline{n})$.

F is well defined since clearly our definition is alternating in the places of both $x_1 \wedge x_2 \wedge \ldots \wedge x_r$ and $y_1 \wedge y_2 \wedge \ldots \wedge y_{n-r}$.

<u>Claim</u> F is non singular.

We compute the matrix of F with respect to the bases $X_r = \{\wedge e_i : i \epsilon \hat{I}(n,r)\}$ of $\Lambda^r E_K$ and $X_{n-r} = \{J(\wedge e_j) : j \epsilon \hat{I}(n,n-r)\}$ of $\Lambda^{n-r} E_K$ (notice that this matrix is square since $|\hat{I}(n,r)| = \binom{n}{r} = \binom{n}{n-r} = |\hat{I}(n,n-r)|$) .

For $\wedge e_i \in X_r$, $\wedge e_j \in X_{n-r}$, $F(\wedge e_i, J(\wedge e_j))$ is the coefficient $\lambda \in K$ appearing in

$$e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r} \wedge e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_r} = \lambda . (e_1 \wedge e_2 \wedge . . \wedge e_n) \quad .$$

This is zero unless $\{i_1, i_2, \ldots, i_r, j_1, j_2, \ldots, j_r\} = \underline{n}$, and for each $i \in \hat{I}(n,r)$ there is precisely one $j \in \hat{I}(n,n-r)$ such that this is so. For this j , $F(\wedge e_i, J(\wedge e_j)) = \pm 1$ and it follows immediately that $(F(\wedge e_i, J(\wedge e_j)))_{i \epsilon \hat{I}(n,r), j \epsilon \hat{I}(n,n-r)}$ is non singular. This completes the proof of the claim.

Let $g \in \Gamma$, then $g^t Jg = J$ implies $g^t J = Jg^{-1}$ and hence, by applying g^t to both sides of (ii), we have

$$g^t x_1 \wedge \ldots \wedge g^t x_r \wedge J(g^{-1} \wedge y_1) \ldots J(g^{-1} \wedge y_{n-r})$$

$$= F(x_1 \wedge \ldots \wedge x_r, y_1 \wedge \ldots \wedge y_{n-r})(\det g) \wedge e_u \quad .$$

Hence $F(x,y) = F(g^t x, g^{-1} y).\det g$ (for all $x \in \Lambda^r E_K$, $y \in \Lambda^{n-r} E_K$, $g \in \Gamma$) ie.

(iii) $$F(x, gy) = F(g^t x, y).\det g$$

for all $x \in \Lambda^r E_K$, $y \in \Lambda^{n-r} E_K$ and $g \in \Gamma$.

This implies that $\Lambda^r E_K \cong (\Lambda^{n-r} E_K)^o \otimes \Lambda^n E_K$ where $(\Lambda^{n-r} E_K)^o$ is the 'w* contravariant' dual of $\Lambda^{n-r} E_K$ ie. for $g \in \Gamma$, $x \in \Lambda^{n-r} E_K$ and $f \in (\Lambda^{n-r} E_K)^*$ (the dual space $\mathrm{Hom}_K(\Lambda^{n-r} E_K, K)$) then

$$g.f(x) = f(w^* g\,(x)) = f(g^t(x))$$

(where we identify $g \in \Gamma$ with its image in $S_K^r(\Gamma)$ then $w^* g = g^t$ (5.1,p.100)).

6.1e Remark

By following through the proof of (6.1d) we can see that the KG isomorphism $\Lambda^r E_K \to \Lambda^{n-r} E_K \otimes \Lambda^n E_K$ maps $\wedge e_i$ $(i \in \bar{I}(n,r))$ to

$\epsilon_{n,r} \wedge e_{i^*} \otimes \wedge e_u$ where $i^* = (i_1^*, i_2^*, \ldots, i_{n-r}^*) \in I(n,n-r)$ is uniquely defined by $\{\bar{i}^*\} = \underline{n} \setminus \{i\}$ and $\epsilon_{n,r} = \pm 1$.

6.1f Corollary

Let $K = Q$, $\lambda = \Lambda^{r,+}(G)$ be permissible with the first column of $[\lambda]$ of length $m > \ell$ and $\lambda' \in \Lambda^{r',+}(G)$ $(r' = r-2m+n)$ be the partition with shape $[\lambda']$ derived from $[\lambda]$ by replacing the first column of $[\lambda]$ with one of length $n-m$.

Then $V^{\Gamma}_{\lambda,Q}$ is isomorphic as a $Q\Gamma$ module to the (irreducible) $Q\Gamma$-module $V^{\Gamma}_{\lambda',Q} \otimes \Lambda^n E_Q$.

Proof

Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r$ be the lengths of the columns of $[\lambda]$ $(\mu_1 = m)$. By (6.1d)

$$\Lambda^{\mu_1} E_Q \otimes \Lambda^{\mu_2} E_Q \otimes \ldots \otimes \Lambda^{\mu_r} E_Q = E^r_Q(C(\lambda))$$

is isomorphic as a $Q\Gamma$ module to

$$\Lambda^{n-\mu_1} E_Q \otimes \Lambda^{\mu_2} E_Q \otimes \ldots \otimes \Lambda^{\mu_r} E_Q \otimes \Lambda^n E_Q = E^{r'}_Q(C(\lambda')) \otimes \Lambda^n E_Q$$

(using appropriate identifications of the tensor products with $E^r_Q(C(\lambda))$ and $E^{r'}_Q(C(\lambda'))$ as in the proof of (5.3d)).

Now, $V^{\Gamma}_{\lambda,Q} \subset E^r_Q(C(\lambda))$ and $V^{\Gamma}_{\lambda',Q} \subset E^{r'}_Q(C(\lambda'))$ so to prove the corollary we need only show that under the isomorphism $V^{\Gamma}_{\lambda,Q}$ is mapped to $V^{\Gamma}_{\lambda',Q} \otimes \Lambda^n E_Q$. But since both are irreducible it is enough to show that one element of $V^{\Gamma}_{\lambda,Q}$ is mapped into $V^{\Gamma}_{\lambda',Q} \otimes \Lambda^n E_Q$.

Consider $f_{h(\lambda)} = e_{h(\lambda)}\{C(\lambda)\} \in V^{\Gamma}_{\lambda,Q}$, then by (6.1e) this is mapped to $\pm\, e_{h(\lambda')}\{C(\lambda')\} \otimes \wedge e_u$ (since the component of $f_{h(\lambda)}$ in $\Lambda^{\mu_1} E_K$ ie. $e_1 \wedge e_2 \wedge \ldots \wedge e_{\mu_1}$, is mapped to $e_1 \wedge e_2 \wedge \ldots \wedge e_{n-\mu_1} \otimes \wedge e_u$). But $e_{h(\lambda')}\{C(\lambda')\} = f_{h(\lambda')} \in V^{\Gamma}_{\lambda',Q}$ and we are finished.

6.1g Remark

Let $\lambda \in \Lambda^+(G)$ be permissible. If, in addition, the first column of $[\lambda]$ is of length $\leq \ell$ we say λ is admissible. Clearly for each admissible λ there is a permissible $\lambda' \in \Lambda^+(\lambda)$ whose first column is of length $> \ell$, which gives rise to λ using the method described in (6.1f) and vice versa.

We return to our study of the spaces $\Lambda^r E_K$.

Let $B: E_K \times E_K \to K$ be the non singular symmetric bilinear form which yields the odd orthogonal group $(n = 2\ell+1)$, then B gives rise to a non singular symmetric bilinear form

6.1h

$$\Lambda^r B: \Lambda^r E_K \times \Lambda^r E_K \to K$$

defined by extending linearly:

$$\begin{aligned} \Lambda^r B(\wedge e_i, \wedge e_j) :&= B^r(e_i, e_j\{G(r)\}) \\ :&= B^r(e_i\{G(r)\}, e_j) \end{aligned}$$

(for all $i,j \in I(n,r)$).

This form is 's^*-contravariant' since $B^r: E_K^r \times E_K^r \to K$ is (4.1b), ie.

$$\Lambda^r B(\phi(\wedge e_i), \wedge e_j) = \Lambda^r B(\wedge e_i, s^*\phi(\wedge e_j))$$

(for all $i,j \in I(n,r)$, $\phi \in S_K^r(G)$) .

Now, let $\alpha \in \Lambda_r(\Gamma)$ then the weight space $\Lambda^r E_K^\alpha$ of $\Lambda^r E_K$, as a $K\Gamma$ module, is the span of those $\wedge e_i$ $(i \in I(n,r))$ with $i \in \lambda \in \Lambda_r(G)$ such that $\lambda^* = \alpha$. It follows that if $\alpha \neq -\beta \in \Lambda_r(\Gamma)$ then $\Lambda^r B(\Lambda^r E_K^\alpha, \Lambda^r E_K^{-\beta}) = 0$ since $\Lambda^r B(\wedge e_i, \wedge e_j) \neq 0$ iff $\bar{i} = j\pi$ (some $\pi \in G(r)$) and if $i \in \lambda$ $(\lambda^* = \alpha)$ then $\bar{i} = (\bar{i}_1, \bar{i}_2, \ldots, \bar{i}_r) \in \lambda' = (\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$ $(\lambda'^* = -\alpha)$.

We can now prove:

6.1i Theorem

For any infinite field K (char $K \neq 2$) and integer $0 \leq r \leq n$, $\Lambda^r E_K$ is an irreducible $K\Gamma$-module.

Proof

By (6.1d) we need only show that each $\Lambda^r E_K$ $(0 \leq r \leq \ell)$ is irreducible. So suppose $0 \leq r \leq \ell$. If $r = 0$ the theorem is certainly true since $\Lambda^0 E_K = K$ by definition, so consider $r > 0$.

We show first that $\Lambda^r E_K$ is generated as a $K\Gamma$ module by $\wedge e_{u_r} := e_1 \wedge e_2 \wedge \ldots \wedge e_r$.

We define the following elements of Γ :

(i) Let $g_\mu \in \Gamma$ $(\mu \in \underline{n})$ be the element such that $g_\mu(e_\mu) = e_{\bar{\mu}}$, $g_\mu(e_{\bar{\mu}}) = e_\mu$ and $g_\mu(e_\nu) = e_\nu$ $(\nu \neq \mu, \bar{\mu})$.

(ii) Let $x_\pi \in \Gamma$ $(\pi \in G(\ell))$ be the element such that $x_\pi(e_\rho) = e_{\pi(\rho)}$ $(\rho \in \underline{\ell})$ (then $x_\pi(e_{\bar{\rho}}) = e_{\overline{\pi(\rho)}}$ $(\rho \in \underline{\ell})$).

(iii) Let $h_\rho \in \Gamma$ $(\rho \in \underline{\ell})$ be the element with 1's down the diagonal, a 1 in the $(\ell+1,\rho)^{th}$ place, a -1 in the $(\bar{\rho},\ell+1)^{th}$ place and zeros elsewhere. Then $h_\rho(e_\rho) = e_{\ell+1}+e_\rho$ and does not change $e_{\rho'}$ $(\rho' \in \underline{\ell}, \rho' \neq \rho)$.

Now, let $i \in I(n,r)$ with $i_1,i_2,\ldots,i_r$ distinct and $i_\rho \neq \ell+1$ $(\rho \in \underline{r})$. Using g_{i_q} to change each e_{i_q} $(i_q \geq \ell+2)$ to $e_{\bar{i}_q}$ (without altering e_{i_p} $(i_p \leq \ell)$) we can assume that $i_1,i_2,\ldots,i_r \leq \ell$. Now, there is some $\pi \in G(\ell)$ such that $x_\pi(\wedge e_{u_r}) = \wedge e_i$. We can now use the h_ρ $(\rho \in \underline{\ell})$ to obtain $\wedge e_j$ with $j_q = \ell+1$ (some $q \in \underline{r}$) from $\wedge e_{u_r}$ and this shows that $\wedge^r E_K$ is generated as $K\Gamma$ module by $\wedge e_{u_r}$ as required.

Now, $u_r = (1,2,\ldots,r) \in I(n,r)$ is of weight $w := 1^r = (1,1,\ldots,1,0,\ldots,0) \in \Lambda_r(G)$ $(r \leq \ell$ 1's) and therefore by (5.2c) and the preamble to this theorem, $\wedge^r E_K^{w^*} = K.\wedge e_{u_r}$. Now, let M be a $K\Gamma$ submodule of $\wedge^r E_K$, then M is a sum $\sum^{\oplus}_{\alpha \in \Lambda_r(\Gamma)} M^\alpha$ of its weight spaces.

Suppose $M^{w^*} \neq (0)$, then $K.\wedge e_{u_r} \subset M$ which implies that $M = \wedge^r E_K$. So assume $M^{w^*} = (0)$, then $M \subset \sum_{\alpha \neq w^*} \wedge^r E^\alpha$. Notice also that $M^{-w^*} = (0)$ since it is easy to see that for any $K\Gamma$ module V and

$\alpha \in \Lambda(\Gamma)$ then $J(V^{\alpha}) = V^{-\alpha}$. Now, let $x \in \Lambda^r E_K$, then $x = \phi \wedge e_{u_r}$ for some $\phi \in S_{r,K}(\Gamma) \subset S_{r,K}(G)$. By (6.1j) $s^*\phi$ is also contained in $S_{r,K}(\Gamma)$ so that for $m \in M$

$$\begin{aligned} \Lambda^r B(x,m) &= \Lambda^r B(\phi \wedge e_{u_r}),m) \\ &= \Lambda^r B(\wedge e_{u_r}, s^*\phi(m)) \\ &= \sum_{\alpha \in \Lambda_r(\Gamma)} \Lambda^r B(\wedge e_{u_r}, m_\alpha) \end{aligned}$$

where $s^*\phi(m) \in M$ is a sum Σm_α $(\alpha \in \Lambda_r(\Gamma))$, $m_\alpha \in M^\alpha$.

But, now $\Lambda^r B(\wedge e_{u_r}, m_\alpha) = 0$ unless $-w^* = \alpha$ and moreover $M^{-w^*} = (0)$, so that $m_{-w^*} = 0$ and hence

$$\Lambda^r B(x,m) = 0 \quad .$$

Thus $\Lambda^r B(x,M) = 0$ for all $x \in \Lambda^r E_K$, but $\Lambda^r B$ is non singular, hence $M = (0)$ and we are finished.

6.1j Remark

In the proof of (6.1i) we have used the fact that if $\phi \in S_{r,K}(\Gamma)$ then $s^*\phi \in S_{r,K}(\Gamma)$. Recall that we have defined $s_\Gamma : K[\Gamma] \to K[\Gamma]$ such that the following diagram commutes:

$$\begin{array}{ccc} K_+[G] & \xrightarrow{s} & K_+[G] \\ {\scriptstyle \Psi_K}\downarrow & & \downarrow{\scriptstyle \Psi_K} \\ K[\Gamma] & \xrightarrow{s_\Gamma} & K[\Gamma] \end{array}$$

(see (2.6g)).

Now, $s_\Gamma(K_r[\Gamma]) = K_r[\Gamma]$ and it therefore induces a map

$$s_\Gamma^* : S_{r,K}(\Gamma) \to S_{r,K}(\Gamma)$$

such that $s^*(\varepsilon_g) = \varepsilon_{\bar{g}}$ where $\bar{g} = (\overline{g_{\mu\nu}}) = (g_{\overline{\nu\mu}}) = J\, g^t\, J = g^{-1}$ $(g \in \Gamma)$ (and $\varepsilon_g \in S_{r,K}(\Gamma)$ is evaluation at g). Thus, s_Γ^* is an antiautomorphism and it is not hard to see that for $\phi \in S_{r,K}(\Gamma) \subset S_{r,K}(G)$

$$s^*(\phi) = s_\Gamma^*(\phi) \quad .$$

6.2 The $Q\Gamma$ modules $D^\Gamma_{\lambda,Q}$

For each permissible $\lambda \in \Lambda^+(G)$ we shall find an irreducible $Q\Gamma$ module in $Q[\Gamma]$, the counterpart of the irreducible QG module $D_{\lambda,Q}$ in $Q_+[G]$.

We deal first with admissible weights. Let $\lambda \in \Lambda^{r,+}(G)(r \geq 0)$ be admissible and define $D^\Gamma_{\lambda,Q}$ to be the image of $D_{\lambda,Q}$ under the restriction map $\Psi_Q : Q_+[G] \to Q[\Gamma]$, (Ψ_Q is a $Q\Gamma$-map, giving $Q_+[G], Q[\Gamma]$ their left $Q\Gamma$ module structure: $g.c^H_{\mu\nu} = \sum_{\rho\in\underline{n}} c^H_{\rho\nu}(g)c^H_{\mu\rho}$ $(g \in H: = G,\Gamma,\ \mu,\nu\in\underline{n}))$.

Define also the $Q\Gamma$-map $\theta^{\Gamma}_{\lambda,Q} : E^{\Gamma}_Q \to D^{\Gamma}_{\lambda,Q}$ to be the composition of $\theta_{\lambda,Q} : E^{\Gamma}_Q \to D_{\lambda,Q}$ (5.1ℓ) with this restriction.

6.2a **Lemma**

For $r \geq 2$ $\operatorname{Im}_Q \beta^{r,+} \subset \ker \theta^{\Gamma}_{\lambda,Q}$.

Proof

Since $\operatorname{Im}_Q \beta^{r,+}$ is spanned by the set $\{\beta^{r,+}_{ab}(e_i) : i \in I(n,r),\ (a,b) \in J_o(r)\}$ it is enough to show each $\beta^{r,+}_{ab}(e_i) \in \ker \theta^{\Gamma}_{\lambda,Q}$.

We have for $i \in I(n,r)$, $(a,b) \in J_o(r)$:

$$\theta_{\lambda,Q}(\beta^{r,+}_{ab}(e_i)) = \theta_{\lambda,Q}(\sum_{\mu \in \underline{n}} e_{i(a,b:\mu,\bar{\mu})})$$

$$= \sum_{\mu \in \underline{n}} (T_h : T_{i(a,b:\mu,\bar{\mu})})$$

where $h = h(\lambda)$.

We show that $\sum_{\mu \in \underline{n}} (T_h : T_{i(a,b:\mu,\bar{\mu})}) \in \ker \Psi_Q$.

$$(T_h : T_{i(a,b:\mu,\bar{\mu})}) = \sum_{\sigma \in C(\lambda)} \varepsilon(\sigma)\, c_{h\sigma,i(a,b:\mu,\bar{\mu})}$$

$$= \sum_{\sigma \in C(\lambda)} \varepsilon(\sigma) c_{k(\sigma),i}\, c_{h_{\sigma(a)}\mu}\, c_{h_{\sigma(b)}\bar{\mu}}$$

where $k(\sigma) = h\sigma[a,b] = h[\sigma(a),\sigma(b)]\sigma_{ab}$ (4.3b).

Hence

$$\sum_{\mu\in\underline{n}}(T_h:T_{i(a,b:\mu,\bar{\mu})}) = \sum_{\sigma\in C(\lambda)}\varepsilon(\sigma)c_{k(\sigma),i}\left(\sum_{\mu\in\underline{n}} c_{h_{\sigma(a)}\mu}c_{h_{\sigma(b)}\bar{\mu}}\right)$$

$$= \sum_{\sigma\in C(\lambda)}\varepsilon(\sigma)c_{k(\sigma),i}(H^G_{h_{\sigma(a)}h_{\sigma(b)}} + \delta_{h_{\sigma(a)}\bar{h}_{\sigma(b)}})$$

But $\Psi_Q(H^G_{h_{\sigma(a)}h_{\sigma(b)}}) = 0$; therefore

$$\Psi_Q\left(\sum_{\mu\in\underline{n}}(T_h:T_{i(a,b:\mu,\bar{\mu})})\right) = \sum_{\sigma\in C(\lambda)}\varepsilon(\sigma)c_{k(\sigma),i}\cdot\delta_{h_{\sigma(a)}\bar{h}_{\sigma(b)}}$$

Now, since λ is admissible, every entry in h (hence also in $h\sigma$) is $\leq \ell$. Therefore $\delta_{h_\rho \bar{h}_{\rho'}} = 0$ for all $\rho,\rho' \in \underline{r}$ and

$$\Psi_Q\left(\sum_{\mu\in\underline{n}}(T_h : T_{i(a,b:\mu,\bar{\mu})})\right) = 0$$

as required.

Denote by $N^\Gamma_{\lambda,Q}$ the kernel of $\theta^\Gamma_{\lambda,Q}$; we have:

<u>6.2b</u> <u>Lemma</u>

The orthogonal complement of $N^\Gamma_{\lambda,Q}$ with respect to the form $B^r : E^r_Q \times E^r_Q \to Q$ is $V^\Gamma_{\lambda,Q} = V_{\lambda,Q} \cap \ker_Q B^r$.

<u>Proof</u>

Clearly $N_{\lambda,Q} + \mathrm{im}_Q B^{r,+} \subset N^\Gamma_{\lambda,Q}$ by (6.2a) and the definition of $D^\Gamma_{\lambda,Q}$. Thus

$$(N_{\lambda,Q} + \mathrm{Im}_Q\beta^{r,+})^{\perp} \supseteq (N^{\Gamma}_{\lambda,Q})^{\perp}$$

the orthogonal complements taken with respect to $B^r : E^r_Q \times E^r_Q \to Q$ (though by (5.1p) we could also take it with respect to $A^r : E^r_Q \times E^r_Q \to Q$) .

But $(N_{\lambda,Q} + \mathrm{Im}_Q\beta^{r,+})^{\perp} = (N_{\lambda,Q})^{\perp} \cap (\mathrm{Im}_Q\beta^{r,+})^{\perp}$

$$= V_{\lambda,Q} \cap \ker_Q\beta^r = V^{\Gamma}_{\lambda,Q}$$

by (4.4f) and (5.1q).

Hence $(N^{\Gamma}_{\lambda,Q})^{\perp} = V^{\Gamma}_{\lambda,Q}$ since $V^{\Gamma}_{\lambda,Q}$ is irreducible (5.3c) and $(N^{\Gamma}_{\lambda,Q})^{\perp} \neq 0$ (since then $N^{\Gamma}_{\lambda,Q} = E^r_Q$ and $D^{\Gamma}_{\lambda,Q} = 0$, but it is easy to see that $\theta_{\lambda,Q}(e_{h(\lambda)}) \notin J_{r,Q}$) . This completes the proof.

We may now define a non-singular bilinear form

6.2c
$$(\ ,\ ) : V^{\Gamma}_{\lambda,Q} \times D^{\Gamma}_{\lambda,Q} \to Q$$

by

$$(x,\theta^{\Gamma}_{\lambda,Q}(y)) : = B^r(x,y)$$

for all $x \in V^{\Gamma}_{\lambda,Q}$, $y \in E^r_Q$.

This form is well defined since $B^r(x,\ker\theta^{\Gamma}_{\lambda,Q}) = 0$ and non singular because if $x \in V_{\lambda,Q}$ with $(x,D^{\Gamma}_{\lambda,Q}) = 0$ then $B^r(x,E^r_Q) = 0$ implies $x = 0$.

Also $(\ ,\ ) : V^{\Gamma}_{\lambda,Q} \times D^{\Gamma}_{\lambda,Q}$ is 's^*-contravariant' ie. for $x \in V^{\Gamma}_{\lambda,Q}$, $z \in D^{\Gamma}_{\lambda,Q}$ and $\phi \in S_{r,Q}(\Gamma)$

$$(\phi(x),z) = (x,s^*_{\Gamma}\phi(z))$$

or

$$(g.x,z) = (x,g^{-1}.z) \qquad (g \in \Gamma)$$

(this follows from the fact that B^{Γ} is s^* contravariant (4.1b) and remark (6.1j)).

This proves that $D^{\Gamma}_{\lambda,Q}$ is isomorphic to the (usual) dual $(V^{\Gamma}_{\lambda,Q})^*$. Hence, it is simple because $V^{\Gamma}_{\lambda,Q}$ is, moreover:

<u>6.2d</u> <u>Proposition</u> $\quad V^{\Gamma}_{\lambda,Q} \cong D^{\Gamma}_{\lambda,Q}$ as $Q\Gamma$-modules.

<u>Proof</u>

For $\alpha \in \Lambda(\Gamma)$ notice that

$$s^*_{\Gamma}(\phi_{\alpha}) = s^*_{\Gamma}\Big(\sum_{\substack{\lambda \in \Lambda_r(G) \\ \lambda^* = \alpha}} \zeta_{\lambda}\Big) = \Sigma\, s^*(\zeta_{\lambda}) = \Sigma\, \zeta_{\lambda^0}$$

where $\lambda^0 = (\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$, and hence $(\lambda^0)^* = -\alpha$ and therefore

$$s^*_{\Gamma}(\phi_{\alpha}) = \phi_{-\alpha} \quad .$$

Now, for any $\alpha, \beta \in \Lambda(\Gamma)$, $x \in V^{\Gamma}_{\lambda,Q}$, $z \in D^{\Gamma}_{\lambda,Q}$ we have:

$$(\phi_\alpha(x),\phi_\beta(z)) = (x, s_\Gamma^*\phi_\alpha.\phi_\beta(z))$$

$$= (x,\phi_{-\alpha}.\phi_\beta(z))$$

$$= 0 \quad \text{unless} \quad -\alpha = \beta \ .$$

It follows that the restriction of $(\ ,\ )$ to $\phi_\alpha V^\Gamma_{\lambda,Q} \times \phi_{-\alpha} D^\Gamma_{\lambda,Q}$ is non degenerate and therefore

$$\dim_Q(V^\Gamma_{\lambda,Q}) = \dim_Q(D^\Gamma_{\lambda,Q})^{-\alpha} \ .$$

But, it is easy to see that

$$J.(D^\Gamma_{\lambda,Q})^{-\alpha} = (D^\Gamma_{\lambda,Q})^\alpha$$

and hence

$$\dim_Q(V^\Gamma_{\lambda,Q})^\alpha = \dim_Q(D^\Gamma_{\lambda,Q})^\alpha \ .$$

Now, the characters of $V^\Gamma_{\lambda,Q}$ and $D^\Gamma_{\lambda,Q}$ are the same and therefore they are isomorphic $Q\Gamma$ modules.

<u>6.2e</u> <u>Remark</u>

The proof above is just a special case of the argument of [Wo,p.42].

Recall that as a KG-module $\Lambda^n E_K$ is isomorphic to $D_{1^n,K} = K.(T_u:T_u) \subset K_+[G]$ (6.1c) and therefore $\Psi_K(D_{1^n,K}) \subset K[\Gamma]$ is isomorphic as a KG-module to $\Lambda^n E_K$. Denote $\Psi_K(D_{1^n,K}) \subset K[\Gamma]$ by $D^{\Gamma}_{1^n,K}$.

Now, let $\lambda \in \Lambda^+(G)$ be permissible with the first column of $[\lambda]$ of length $> \ell$ and $\lambda' \in \Lambda^+(G)$ be its associated admissible weight.

It follows from (6.1f), (6.2d) and the remarks above that

$$V^{\Gamma}_{\lambda,Q} \cong V^{\Gamma}_{\lambda',Q} \otimes \Lambda^n E_Q \cong D^{\Gamma}_{\lambda',Q} \otimes D^{\Gamma}_{1^n,Q}$$

as $Q\Gamma$-modules.

Further, we can clearly identify $D^{\Gamma}_{\lambda',Q} \otimes D^{\Gamma}_{1^n,Q}$ with the $Q\Gamma$-submodule

$$\{x.y \in Q[\Gamma] : x \in D^{\Gamma}_{\lambda',Q}, y \in D^{\Gamma}_{1^n,Q}\}$$

of $Q[\Gamma]$. Denote this $Q\Gamma$ submodule by $D^{\Gamma}_{\lambda,Q}$.

Now, it is not hard to see that we can define a s^* contravariant non singular bilinear form

$$(\ ,\ ) : V^{\Gamma}_{\lambda,Q} \otimes D^{\Gamma}_{\lambda,Q} \to Q$$

using the identifications above and the form related to admissible λ' .

To complete the picture we have

<u>6.2f</u> <u>Proposition</u>

$\lambda \in \Lambda^+(G)$ permissible then

$$D^{\Gamma}_{\lambda,Q} = \Psi_Q(D_{\lambda,Q}) \quad .$$

<u>Proof</u>

If λ is admissible this is just a definition, so suppose λ is not admissible and let λ' be its associated admissible weight in $\Lambda^+(G)$.

Now, the $K\Gamma$ isomorphism (6.1e) $\Lambda^m E_K \to \Lambda^{n-m} E_K \otimes \Lambda^n E_K$ $(m \leq n)$ takes $\Lambda e_i \to \varepsilon_{n,m} \Lambda e_{i*} \otimes \Lambda e_u$ where $i_1 < i_2 < \ldots < i_m$, $i_1^* < i_2^* < \ldots < i_{n-m}^*$ and $\{i_1, i_2, \ldots, i_m, i_1^*, i_2^*, \ldots, i_{n-m}^*\} = \underline{n}$ and $\varepsilon_{n,m} = \pm 1$ depends only on n and m . In particular if $i = (1,2,\ldots,m)$ then $i^* = (1,2,\ldots,n-m)$ so for any $g \in \Gamma$ the coefficient of $e_1 \wedge e_2 \wedge \ldots \wedge e_m$ in $g(\wedge e_j)$ $(j \in I(n,m))$ is the same as that of $e_1 \wedge e_2 \wedge \ldots \wedge e_{n.m}$ in $\varepsilon_{n,m} . \det g . g(\wedge e_{j*})$. Hence

$$\begin{vmatrix} g_{1,i_1} & g_{1,i_2} \cdots & g_{1,i_m} \\ g_{2,i} & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ g_{m,i_1} & \cdots\cdots & g_{m,i_m} \end{vmatrix} = \varepsilon_{n,m} . \det g \begin{vmatrix} g_{1,i_1^*} & g_{1,i_2^*} \cdots & g_{1,i_{n-m}^*} \\ g_{2,i_1^*} & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ g_{n-m,i_1^*} & \cdots\cdots & g_{n-m,i_{n-m}^*} \end{vmatrix}$$

It now follows immediately from the definition of bideterminant (5.1h) that for any $i \in I(n,r)$ $(|\lambda| = r)$ there is some $j \in I(n,r')$ $(|\lambda'| = r')$ such that for all infinite fields K:

$$\Psi_K((T_{h(\lambda)} : T_i) - \varepsilon_{n,m} \det(C)(T_{h(\lambda')} : T_j)) = 0$$

where $C = (c_{\mu\nu})_{\mu,\nu\in\underline{n}}$ and m is the length of the first column of $[\lambda]$.

Hence $\Psi_K(D_{\lambda,K}) = \Psi_K((T_u:T_u))\Psi_K(D_{\lambda',K})$ with $u = (1,2,..,n)$ and by letting $K = Q$ we are finished.

6.3 A Z-form of $D^{\Gamma}_{\lambda,Q}$

In view of the results of the last section we may as well consider only those permissible $\lambda \in \Lambda^+(G)$ which are also admissible since any Z-form we have for $D^{\Gamma}_{\lambda,Q}$ when tensored with the determinant function will certainly give a Z-form of $D^{\Gamma}_{\lambda',Q}$, where λ' is the permissible weight associated with λ.

Let $\lambda \in \Lambda^{\Gamma,+}(G)$ be admissible, we have the s.e.s. of $Q\Gamma$-modules:

6.3a

$$0 \to M_{\lambda,Q} \to E^{\Gamma}_Q \xrightarrow{\theta^{\Gamma}_{\lambda,Q}} D^{\Gamma}_{\lambda,Q} \to 0$$

where $M_{\lambda,Q} = (\mathrm{Im}_Q\beta^+ + N_{\lambda,Q})$.

By restricting $\theta^{\Gamma}_{\lambda,Q}$ to E^{Γ}_Z we have the s.e.s. of Z-modules:

6.3b

$$0 \to M_{\lambda,Z} \to E^{\Gamma}_Z \xrightarrow{\theta^{\Gamma}_{\lambda,Q}} D^{\Gamma}_{\lambda,Z} \to 0$$

where $D^{\Gamma}_{\lambda,Z}$ is the image of E^{Γ}_Z under $\theta^{\Gamma}_{\lambda,Q}$ and $M_{\lambda,Z} = M_{\lambda,Q} \cap E^{\Gamma}_Z$.

Now, $D^{\Gamma}_{\lambda,Z} \subset Q[\Gamma]$ is clearly torsion free and finitely generated as a Z-module, it is therefore a free Z-module. Hence, for any infinite field K (char $K \neq 2$) we have a s.e.s. of $K\Gamma$-modules:

6.3c

$$0 \to K \otimes M_{\lambda,Z} \to K \otimes E_Z^r \xrightarrow{1\otimes\theta^\Gamma_{\lambda,Q}} K \otimes D^\Gamma_{\lambda,Z} \to 0$$

where the action of $K\Gamma$ is given by (3.2a).

For $K = Q$ it is clear that $Q \otimes D^\Gamma_{\lambda,Z}$ is isomorphic as $Q\Gamma$-module to $D^\Gamma_{\lambda,Q}$ thus $\dim_K D^\Gamma_{\lambda,Z} \otimes K = \dim_Q D^\Gamma_{\lambda,Q}$ = rank of $D^\Gamma_{\lambda,Z}$ as a free Z-module, and therefore $D^\Gamma_{\lambda,Z}$ is an admissible Z-form of $D^\Gamma_{\lambda,Q}$. (clearly $cf(D^\Gamma_{\lambda,Z}) \subset Z[\Gamma]$) (3.2,p.64).

Now, E_K^r is naturally isomorphic as a $K\Gamma$-module to $K \otimes E_Z^r$ via the $K\Gamma$-map $e_i^K \to 1 \otimes e_i \in K \otimes E_Z^r$ $(i \in I(n,r)$, $e_i^K := e_i \in E_K^r)$ and we may therefore identify $K \otimes M_{\lambda,Z}$ with a $K\Gamma$-submodule $M_{\lambda,K} \subset E_K^r$. Our aim is to find a spanning set for this submodule.

When $K = Q$ we know that $M_{\lambda,Q} = \mathrm{Im}_Q\beta^+ + N_{\lambda,Q}$ and therefore we may well ask the question: is $M_{\lambda,K} = (\mathrm{Im}_K\beta^+ + N_{\lambda,K})$?

Let $\overline{M}_{\lambda,K} = (\mathrm{Im}_K\beta^+ + N_{\lambda,K})$ then we have a s.e.s. of $K\Gamma$-modules.

6.3d

$$0 \to \overline{M}_{\lambda,K} \to E_K^r \xrightarrow{\overline{\theta}_{\lambda,K}} \overline{D}_{\lambda,K} \to 0$$

where $\overline{D}_{\lambda,K} := E_K^r/\overline{M}_{\lambda,K}$, and $\overline{\theta}_{\lambda,K}$ is the natural map.

Now, $\overline{M}_{\lambda,K}$ is spanned by the set of elements

$$X_K = \{\beta_{ab}^{r,+}(e_i),\ R_1, R_2, R_3 \colon (a,b) \in J(r),\ i \in I(n,r)\}$$

of E_K^r, where R_1, R_2 and R_3 are the elements spanning $N_{\lambda,K}$ given in (5.1m).

Hence, with $K = Q$ we see that $M_{\lambda,Z}$ contains the set X_Q and moreover it is easy to see that X_K (any K) is mapped to $1_K \otimes X_Q = \{1_K \otimes x : x \in X_Q\}$ $K \otimes E_Z^r$ under the $K\Gamma$ isomorphism $\alpha : E_K^r \to K \otimes E_Z^r$, thus $\bar{M}_{\lambda,K} \subseteq M_{\lambda,K}$. We now have the following commutative diagram

6.3e
$$\begin{array}{ccccc} K \otimes M_{\lambda,Z} & \to & K \otimes E_Z^r & \xrightarrow{1\otimes\theta^\Gamma_{\lambda,Q}} & K \otimes D^\Gamma_{\lambda,Z} \\ \uparrow i & & \uparrow \wr \alpha & & \uparrow j \\ \bar{M}_{\lambda,K} & \to & E_K^r & \xrightarrow{\bar{\theta}_{\lambda,K}} & \bar{D}_{\lambda,K} \end{array}$$

where i is the $K\Gamma$-monomorphism induced by α and $j:\bar{D}_{\lambda,K} \to K \otimes D^\Gamma_{\lambda,Z}$ is the $K\Gamma$-epimorphism given by:

$$j(\bar{\theta}_{\lambda,K}(x)) = 1 \otimes \theta^\Gamma_{\lambda,Q}(\alpha(x)) \qquad (x \in E_K^r) .$$

It is evident that $\bar{M}_{\lambda,K}$ may not be 'large' enough to make j an isomorphism. In fact, unless the characteristic of K is zero or large, the work of Lancaster and Towber [LT] implies that we must add more elements to $\bar{M}_{\lambda,K}$. In the next section we construct the likely candidates.

6.4 Generalised Weyl operators

In (4.3) we defined $K\Gamma$-maps $\beta^r_{ab}, \beta^{r,+}_{ab}$ $(r \geq 2, (a,b) \in J(r))$

following Weyl's work in characteristic zero. Here, we define 'generalised Weyl operators', functions on tensor products of the $K\Gamma$-modules $\Lambda^r E_K$ $(r \geq 2)$.

Let K be any field, $B: E_K \times E_K \to K$ any non singular symmetric bilinear form and Γ_B the subgroup of G it defines.

<u>6.4a</u>

For $f \geq 2$, $r \geq f$, let $a_1, a_2, \ldots, a_f \in \underline{r}$ be distinct. Define $[a_1, a_2, \ldots, a_f] : \underline{r-f} \to \underline{r}$ to be the unique order preserving map such that $\mathrm{Im}[a_1, a_2, \ldots, a_f] = \underline{r} \setminus \{a_1, a_2, \ldots, a_f\}$. Clearly when $f = 2$ this coincides with our previous definition (4.3a). We also have the corresponding result to (4.3b):

<u>6.4b</u>

Let $r \geq f$ and $\pi \in G(r)$ then there exists a unique $\pi_{a_1, a_2, \ldots, a_f} \in G(r-f)$ such that the following diagram commutes:

$$\begin{array}{ccc}
\underline{r-f} & \xrightarrow{[a_1, a_2, \ldots, a_f]} & \underline{r} \\
{\scriptstyle \pi_{a_1, a_2, \ldots, a_f}} \downarrow & & \downarrow {\scriptstyle \pi} \\
\underline{r-f} & \xrightarrow[{[\pi(a_1), \pi(a_2), \ldots, \pi(a_f)]}]{} & \underline{r}
\end{array}$$

As before, for $i \in I(n,r)$ we write $i[a_1, a_2, \ldots, a_f] : \underline{r-f} \to \underline{r}$ for the composition of $[a_1, a_2, \ldots, a_f]$ with i, then $i[a_1, a_2, \ldots, a_f]$ is identified with an element of $I(n,r-f)$. Denote by i_a the element $(i_{a_1}, i_{a_2}, \ldots, i_{a_f}) \in I(n,f)$. Finally, write $G(a_1, a_2, \ldots, a_f)$ for the subgroup $\{\pi \in G(r) : \pi(a_1, a_2, \ldots, a_f) = (a_1, a_2, \ldots, a_f)\}$ of $G(r)$.

Let $f \geq 2$, $r,s \geq f$, $a = \{a_1, a_2, \ldots, a_f\} \subseteq \underline{r}$ (a_p's distinct) and $b = \{b_1, b_2, \ldots, b_f\} \subseteq \underline{s}$ (b_p's distinct). For $i \in I(n,r)$, $h \in I(n,s)$ define $F(i,h) \in \Lambda^{r-f}E_K \otimes \Lambda^{s-f}E_K$ by:

<u>6.4c</u>

$$F(i,h) = \Lambda^f B(\wedge e_{i_a}, \wedge e_{h_i}) \wedge e_{i[a]} \otimes \wedge e_{h[b]}$$

where $\Lambda^f B : \Lambda^f E_K \times \Lambda^f E_K \to K$ is the non singular symmetric bilinear form induced by $B : E_K \times E_K \to K$.

<u>6.4d</u> <u>Lemma</u>

For $\pi \in G(a) \subseteq G(r)$ and $\tau \in G(b) \subseteq G(s)$, then

$$F(i\pi, h\tau) = \epsilon(\pi)\epsilon(\tau)F(i,h) \quad .$$

<u>Proof</u>

Since π fixes $a = \{a_1, a_2, \ldots, a_f\}$ it factorizes into two parts π_a and π^a where $\pi_a \in G(a)$ is the identity outside a and π^a is the identity on a , then $\pi = \pi_a \pi^a = \pi^a \pi_a$. Clearly, we can identify π^a , when necessary, with an element of $G(r-f)$ such that $\pi[a] = [a]\pi^a : \underline{r-f} \to \underline{r-f}$. Similarily $\tau = \tau_b \tau^b = \tau^b \tau_b$ and τ^b can be identified, when necessary, with an element of $G(s-f)$ such that $\tau[b] = [b]\tau^b : \underline{s-f} \to \underline{s-f}$.

Now, $\wedge e_{i\pi[a]} \otimes \wedge e_{h\tau[b]} = \wedge e_{i[a]\pi^a} \otimes \wedge e_{h[b]\tau^b} =$

$\varepsilon(\pi^a)\varepsilon(\tau^b)(\wedge e_{i[a]} \otimes \wedge e_{h[b]})$ and also

$\Lambda^f B(\wedge e_{i_{\pi(a)}}, \wedge e_{h_{\pi(b)}}) = \Lambda^f B(\wedge e_{i_{\pi_a(a)}} \otimes \wedge e_{h_{\tau_b(b)}}) =$

$\Lambda^f B(\wedge e_{i_a\cdot\pi_a}, \wedge e_{h_b\cdot\tau_b}) = \varepsilon(\pi_a)\varepsilon(\pi_b)\Lambda^f B(\wedge e_{i_a}, \wedge e_{h_b})$.

Thus, $F(i\pi,h\tau) = \varepsilon(\pi_a)\varepsilon(\pi^a)\varepsilon(\tau_b)\varepsilon(\tau^b)F(i,h) = \varepsilon(\pi)\varepsilon(\tau)F(i,h)$ as required.

We can now define a $K\Gamma_B$ map $\beta^{rs}_{ab} : \Lambda^r E_K \otimes \Lambda^s E_K \to \Lambda^{r-f} E_K \otimes \Lambda^{s-f} E_K$ as follows: for each $i \in I(n,r)$, $h \in I(n,s)$ let

6.4e

$$\beta^{rs}_{ab}(\wedge e_i \otimes \wedge e_h) = \sum_{(\pi,\tau)} \varepsilon(\pi)\varepsilon(\tau)F(i\pi,h\tau)$$

where $(\pi,\tau) \in G(r) \times G(s)$ run over a transversal for the set of right cosets of $G(a) \times G(b)$ in $G(r) \times G(s)$.

This map is well defined and by (6.4d) does not depend on the choice of transversal of $G(a) \times G(b)$ in $G(r) \times G(s)$. Clearly, for $g \in \Gamma_B$, $g\beta^{rs}_{ab} = \beta^{rs}_{ab}g$. so that it is a $K\Gamma_B$ map.

6.4f Remark

Let $\sigma \in G(r) \times G(s)$ then $\sigma(a,b) = (c,d) \in \underline{r} \times \underline{s}$ and $\sigma.G(a) \times G(b)$ is precisely the set of all elements of $G(r) \times G(s)$ which take (a,b) to (c,d) . Conversely, if c,d are subsets of $\underline{r}$, $\underline{s}$ of size f then there is some $\sigma \in G(r) \times G(s)$ such that $\sigma(a,b) = (c,d)$, we define such an element $\sigma_{(c,d)} = (\pi_{(c)}, \tau_{(d)})$ as follows: assume, without loss of generality, that

(i) $c = \{c_1, c_2, \ldots, c_f\}$, $\underline{r} \setminus c = \{c_{f+1}, c_{f+2}, \ldots, c_{r-f}\}$ with $c_1 < c_2 < \ldots < c_f$, and $c_{f+1} < c_{f+2} < \ldots < c_{r-f}$

(ii) $d = \{d_1, d_2, \ldots, d_f\}$, $\underline{s} \setminus d = \{d_{f+1}, d_{f+2}, \ldots, d_{s-f}\}$ with $d_1 < d_2 < \ldots < d_f$ and $d_{f+1} < d_{f+2} < \ldots < d_{s-f}$

(iii) $\underline{r} \setminus \underline{a} = \{a_{f+1}, a_{f+2}, \ldots, a_{r-f}\}$, $\underline{s} \setminus b = \{b_{f+1}, b_{f+2}, \ldots, b_{s-f}\}$ in some fixed order and define $\pi_{(c)} \in G(r)$ by $\pi_{(c)}(a_\rho) = c_\rho$ $(\rho \in \underline{r})$ and $\tau_{(d)} \in G(r)$ by $\tau_{(d)}(b_\nu) = d_\nu$ $(\nu \in \underline{s})$.

We can now rewrite (6.4e) as:

6.4g

$$\beta^{rs}_{ab}(\Lambda e_i \otimes \Lambda e_h) = \sum_{\substack{c \subseteq \underline{r} \\ d \subseteq \underline{s} \\ |c| = |d| = f}} \varepsilon(\sigma_{(c,d)}) \Lambda^f B(\Lambda e_{i_c} \Lambda e_{h_d}) \Lambda e_{i[c]} \otimes \Lambda e_{h[d]} \,.$$

6.4h Remark

When $|f| = 1$ we have the following commutative diagram

$$\begin{array}{ccc} E_K^{r+s} & \xrightarrow{\beta^{r+s}_{a,r+b}} & E_K^{r} \\ \wedge \downarrow & & \downarrow \wedge \\ \Lambda^r E_K \otimes \Lambda^s E_K & \xrightarrow{\beta^{rs}_{a,b}} & \Lambda^{r-1} E_K \otimes \Lambda^{s-1} E_K \end{array}$$

where the two vertical maps are those given by mapping a basis element

$e_i \otimes e_j \in E^{\rho+\rho'}$ to $\wedge e_i \otimes \wedge e_j \in \Lambda^{\rho} E_K \otimes \Lambda^{\rho'} E_K$ $(\rho, \rho' \geq 0)$.

In fact $\beta^{rs}_{a,b} \circ \wedge = \pm (\wedge \circ \beta^{r+s}_{c,d})$ for any $c \in \underline{r}$, $d \in \{r+1, r+2, \ldots, r+s\}$ and $\wedge \circ \beta^{r+s}_{c,d} = 0$ if $c,d \in \underline{r}$ or $c,d \in \{r+1, r+2, \ldots, r+s\}$. Thus, the 'generalised Weyl operator' $\beta^{rs}_{a,b}$ is a true generalisation of the maps of (4.3).

6.4i

We define the Kr_B map

$$\beta^{rs,+}_{a,b} \; \Lambda^{r-f} E_K \otimes \Lambda^{s-f} E_K \to \Lambda^r E_K \otimes \Lambda^s E_K$$

to be the map dual to $\beta^{rs}_{a,b}$ ie.

$$\Lambda^{r,s} B(\beta^{rs,+}_{a,b}(x) y) = \Lambda^{r-f,s-f} B(x, \beta^{rs}_{a,b}(y))$$

for all $x \in \Lambda^{r-f} E_K \otimes \Lambda^{s-f} E_K$ and $y \in \Lambda^r E_K \otimes \Lambda^s E_K$ where for any $\rho, \rho' \geq 0$ $\Lambda^{\rho,\rho'} B : \Lambda^{\rho} E_K \otimes \Lambda^{\rho'} E_K \times \Lambda^{\rho} E \otimes \Lambda^{\rho'} E_K \to K$ is the non singular symmetric bilinear form induced by $B : E_K \times E_K \to K$ so that for $i,j \in I(n,\rho)$ $h,k \in I(n,\rho')$

$$\Lambda^{\rho,\rho'} B(\wedge e_i \otimes \wedge e_h, \wedge e_j \otimes \wedge e_k) := \Lambda^{\rho} B(\wedge e_i, \wedge e_j) \Lambda^{\rho'} B(\wedge e_h, \wedge e_k) \quad .$$

6.4j Lemma

For $i \in I(n, r-f)$, $h \in I(n, s-f)$:

$$\beta^{rs,+}_{a,b}(\wedge e_i \otimes \wedge e_h) = \sum_{\substack{1 \le j_1 < j_2 < \ldots < j_f \le n \\ 1 \le k_1 < k_2 < \ldots < k_f \le n}} \wedge^f B(\wedge e_j, \wedge e_k) \wedge e_{i(a:j)} \otimes \wedge e_{h(b:k)}$$

where $i(a:j) \in I(n,r)$ is the unique element such that $i(a:j)_{a_\rho} = j_\rho$ $(\rho \in \underline{f})$ and $i(a:j)[a] = i$, and $h(b:k) \in I(n,s)$ is the unique element such that $h(b:k)_{b_\rho} = k_\rho$ $(\rho \in \underline{f})$ and $h(b:k)[b] = h$.

Proof

The set X of elements

$$\{\wedge e_i \otimes \wedge e_h : 1 \le i_1 < i_2 < \ldots < i_{r-f} \le n, 1 \le h_1 < h_2 < \ldots < h_{s-f} \le n\}$$

is a basis of $\wedge^{r-f} E_K \otimes \wedge^{s-f} E_K$ and the set Y of elements $\wedge e_{i(a:j)} \otimes \wedge e_{h(b:k)}$ which are non zero, where $1 \le j_1 < j_2 < \ldots < j_f \le n$, $1 \le k_1 < k_2 < \ldots < k_f \le n$ and $\wedge e_i \otimes \wedge e_h \in X$, is a basis of $\wedge^r E_K \otimes \wedge^s E_K$. Thus, it is enough to show that, for each $x \in X$, $y \in Y$

$$\wedge^{r,s} B(\beta^{rs,+}_{a,b}(x), y) = \wedge^{r-f,s-f} B(x, \beta^{rs}_{a,b}(y)) \quad .$$

Let $x = \wedge e_i \otimes \wedge e_h \in X$, $y = \wedge e_{p(a:j)} \otimes \wedge e_{q(b:k)} \in Y$, then $\wedge^{r,s} B(\beta^{rs,+}_{a,b}(x), y) = 0$ unless there is some $\wedge e_{i(a:j')} \otimes \wedge e_{h(b:k')} \in Y$ with $i(a:j')\pi_{(c)} = p(a:j)$ (some $c \subset \underline{r}$, $|c| = f$) and $h(b:k')\tau_{(d)} = q(b:k)$ (some $d \subset \underline{s}$, $|d| = f$) when it is equal to

$$\varepsilon(\pi_{(c)})\varepsilon(\tau_{(d)}) \wedge^f B(\wedge e_{j'}, \wedge e_{k'}) \qquad \ldots\ldots. \qquad (A)$$

Notice that $j'_\rho = p(a:j)_{\pi(c)}(a_\rho)$, $k'_\rho = q(b:k)_{\tau(d)}(b_\rho)$, $(\rho\epsilon\underline{f})$.

Now, $\Lambda^{r,s}B(x,\beta^{rs}_{a,b}(y)) = 0$ unless there is some $\bar{c} \subset \underline{r}$, $|\bar{c}| = f$ and $\bar{d} \subset \underline{s}$, $|\bar{d}| = f$ such that $p(a:j)[\bar{c}] = i$ and $q(b:k)[\bar{d}] = h$ when it is then equal to

$$\epsilon(\sigma_{(\bar{c},\bar{a})})\Lambda^f B(\Lambda e_j,\ldots,\Lambda e_k,\ldots) \qquad \ldots\ldots \qquad (B)$$

with $j''_\rho = p(a:j)_{\pi(\bar{c})}(a_\rho)$, $k''_\rho = q(b:k)_{\tau(\bar{d})}(b_\rho)$, $(\rho\epsilon\underline{f})$ since $\sigma_{(\bar{c},\bar{d})} = (\pi_{(\bar{c})},\tau_{(\bar{d})})$.

But $c = \bar{c}$, $d = \bar{d}$ since otherwise $p(a:j)$ $(q(b:k))$, which contains i (h) as some of its entries, would contain a repetition, i_ρ (h_ρ) and then $\Lambda e_{p(a:j)} = 0$ $(\Lambda e_{q(b:k)} = 0)$, a contradiction.

Thus (A) = (B) , since $\epsilon(\sigma_{(c,d)}) = \epsilon(\pi_{(c)})\epsilon(\tau_{(d)})$.

6.4k Remark

(i) By (6.4h), when $f = 1$ we have $\beta^{rs,+}_{a,b} \circ \Lambda = \pm(\Lambda \circ \beta^{r+s,+}_{c,d})$ for any $c \in \underline{r}$, $d \in \{r+1,r+2,\ldots,r+s\}$ and since $\Lambda \circ \beta^{r+s,+}_{c,d} = 0$ if $c,d \in \underline{r}$, $c,d \in \{r+1,r+2,\ldots,r+s\}$ our definition of $\beta^{rs,+}_{a,b}$ is a true generalisation of the maps of (4.4).

(ii) Let $K = Q$, then it is easy to see that $\beta^{rs,+}_{a,b} \circ \Lambda(e_i \otimes e_h)$ $(i \in I(n,r-f),\ h \in I(n,s-f))$ equals:

$$\pm \frac{1}{f!}\cdot\beta^{r+s,+}_{c,d}\Big\{\sum_{\substack{\mu_\sigma \epsilon \underline{n} \\ \sigma\epsilon\underline{f-1}}} e_{\mu_1}\otimes e_{\mu_2}\otimes\ldots\otimes e_{\mu_{f-1}}\otimes e_i\otimes e_{\bar{\mu}_{f-1}}\otimes\ldots\otimes e_{\bar{\mu}_2}\otimes e_{\bar{\mu}_1}\otimes e_h\Big\}$$

Thus $\mathrm{Im}\beta^{rs,+}_{a,b} \subseteq \wedge \circ \mathrm{Im}\beta^{r+s,+}$ when char $K = 0$.

The proofs of the following two lemmas follow exactly those of their counterparts (4.4f) and (4.5e) respectively.

6.4ℓ Lemma

$(\mathrm{Im}_K\beta^{rs,+}_{a,b})^{\perp} = \ker_K\beta^{rs}_{ab}$ where the orthogonal complement is taken with respect to the form $\Lambda^{r,s}B$ on $\Lambda^r E_K \otimes \Lambda^s E_K$.

6.4m Lemma

Let K be algebraically closed. Consider the form $A:E_K \times E_K \to K$ (5.1,p.100). By elementary linear algebra there is some non singular map $P:E_K \to E_K$ such that $B(Px,Py) = A(x,y)$ for all $x,y \in E_K$. For $f \geq 2$, $r,s \geq f$ and $a \subseteq \underline{r}$, $b \subseteq \underline{s}$ with $|a| = |b| = f$, denote by γ^{rs}_{ab} and $\gamma^{rs,+}_{ab}$ the Kr_A maps constructed using A as β^{rs}_{ab} and $\beta^{rs,+}_{ab}$ respectively, were constructed using B . Then

$$\mathrm{Im}_K\beta^{rs,+}_{ab} = P^{r,s}\mathrm{Im}_K\gamma^{rs,+}_{ab}$$

$$\ker_K\beta^{rs}_{ab} = P^{r,s}\ker_K\gamma^{rs}_{ab}$$

where $P^{r,s}:\Lambda^r E_K \otimes \Lambda^s E_K \to \Lambda^r E_K \otimes \Lambda^s E_K$ is the nonsingular map induced by $P:E_K \to E_K$.

6.5 Modular Reduction of $D^{\Gamma}_{\lambda,Q}$

Let K be an infinite field (char $K \neq 2$) and $B:E_K \times E_K \to K$ be the non singular symmetric bilinear form yielding $\Gamma = O_{2\ell+1}(K)$.

Let $\lambda\epsilon\Lambda^{r,+}(G)$ $(r \geq 0)$ and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r$ be the lengths of the columns of $[\lambda]$. Recall that we can identify $E_K^r(C(\lambda))$ with the tensor product $E_K^{\mu_1}(G(\mu_1)) \otimes E_K^{\mu_2}(G(\mu_2)) \otimes .. \otimes E_K^{\mu_r}(G(\mu_r))$ where $G(\mu_\rho)$ $(\rho\epsilon\underline{r})$ acts on elements of the ρ^{th} column T_ρ^λ of T^λ. (If $\mu_\rho = 0, G(\mu_\rho)$ is empty, $E_K^0(G(0)) := K$), (proof of (5.3d).) Using this identification we can define a non singular symmetric bilinear form:

6.5a

$$\Lambda^\lambda B : E_K^r(C(\lambda)) \times E_K^r(C(\lambda)) \to K$$

by extending linearly to the whole of $E_K^r(C(\lambda)) \times E_K^r((\lambda))$ the map

$$\Lambda^\lambda B(x_1 \otimes x_2 \otimes .. \otimes x_r, y_1 \otimes y_2 \otimes .. \otimes y_r) = \prod_{\rho\epsilon\underline{r}} \Lambda^{\mu_\rho} B(x_\rho, y_\rho)$$

where $x_\rho, y_\rho \in E^{\mu_\rho}(G(\mu_\rho))$ $(\rho\epsilon\underline{r})$.

Let $\rho, \rho' \in \underline{r}$ with $\rho < \rho'$. For subsets a of T_ρ^λ , b of $T_{\rho'}^\lambda$, with $|a| = |b| = f$, we can define, using the results of the last section, $K\Gamma$-maps

6.5b

$$\theta_{ab}^\lambda : E_K^r(C(\lambda)) \to E^{r-2f}(C(\lambda(a,b:f)))$$

and

6.5c

$$\beta_{ab}^{\lambda,+} : E_K^{r-2f}(C(\lambda(a,b:f)) \to E_K^r(C(\lambda))$$

(where $C(\lambda(a,b:f)) = G(\mu_1)\times..\times G(\mu_\rho - f)\times..\times G(\mu_{\rho'}-f)\times..\times G(\mu_r))$ as follows:

Let $x_\rho \in E^{\mu_\rho}(G(\mu_\rho))$ $(\rho\epsilon\underline{r})$ then (using the identification above) extend linearly to the whole of $E_K^r(C(\lambda))$ the map

6.5d

$$\beta_{ab}^{\lambda}(x_1 \otimes x_2 \otimes \ldots \otimes x_r) = \sum_{(1),(2)} x_1 \otimes \ldots \otimes y_\rho^{(1)} \otimes \ldots \otimes y_{\rho'}^{(2)} \otimes \ldots \otimes x_r$$

where $\beta_{ab}^{\mu_\rho,\mu_{\rho'}}(x_\rho \otimes x_{\rho'}) = \sum_{(1),(2)} y_\rho^{(1)} \otimes y_{\rho'}^{(2)}$ $(y_\rho^{(1)} \in E_K^{\mu_\rho - f}(G(\mu_\rho - f))$, $y_{\rho'}^{(2)} \in E_K^{\mu_{\rho'}-f}(G(\mu_{\rho'}-f)))$.

Similarily, for $y_\rho \in E_K^{\mu_\rho - f}(G(\mu_\rho - f)), y_{\rho'} \in E_K^{\mu_{\rho'}-f}(G(\mu_{\rho'}-f))$.

6.5e

$$\beta_{ab}^{\lambda,+}(x_1 \otimes x_2 \otimes ..\otimes y_\rho \otimes ..\otimes y_{\rho'} \otimes ..\otimes x_r) = \sum_{(1),(2)} x_1 \otimes ..\otimes x_\rho^{(1)} \otimes ..\otimes x_{\rho'}^{(2)} \otimes ..\otimes x_r$$

where $\beta_{ab}^{\mu_\rho,\mu_{\rho'}}(y_\rho \otimes y_{\rho'}) = \sum_{(1),(2)} x_\rho^{(1)} \otimes x_{\rho'}^{(2)}$ $(x_\rho^{(1)} \in E_K^{\mu_\rho}(G(\mu_\rho))$

$x_{\rho'}^{(2)} \in E_K^{\mu_\rho}(G(\mu_{\rho'})))$, and extend linearly.

It follows, from the definition (6.5a) of $\Lambda^\lambda B$, that $\beta_{ab}^{\lambda,+}$ is the map dual to β_{ab}^{λ} .

6.5f

Recall (5.1m), that the kernel $N_{\lambda,K}$ of the G-map $\theta_{\lambda,K} : E_K^r \to D_{\lambda,K}$ is spanned by the set of elements $R_1 \cup R_2 \cup R_3$ where:

(i) R_1 consists of all e_i $(i \in I(n,r))$ such that T_i has a repeated entry in two distinct places of some column.

(ii) R_2 consists of all $e_i - \varepsilon(\sigma)e_{i\sigma}$ $(i \in I(n,r), \sigma \in C(\lambda))$.

(iii) $R_3 :=$ the Garnir relations.

Now, consider the map

$$w_K^\lambda : E_K^r \to E_K^r\{C(\lambda)\}$$
$$e_i \to e_i\{C(\lambda)\} \qquad (i \in I(n,r)) .$$

It is not hard to show that the kernel of w_K^λ is spanned by the elements $R_1 \cup R_2$. Thus, we can factor $\theta_{\lambda,K}$ through $E_K^r\{C(\lambda)\}$ giving a s.e.s.

$$\dot{N}_{\lambda,K} \to E_K^r\{C(\lambda)\} \xrightarrow{\dot{\theta}_{\lambda,K}} D_{\lambda,K}$$

where $\dot{\theta}_{\lambda,K}$ is the KG map induced by $\theta_{\lambda,K}$ and $\dot{N}_{\lambda,K} = w_K^\lambda(N_{\lambda,K})$ is spanned by the set of elements $\dot{R}_3 = \{x.\{C(\lambda)\} : x \in R_3\}$.

6.5g

Let $K = Q$ and $\lambda \in \Lambda^{r,+}(G)$ be admissible. Recall that we have defined the Qr map $\theta_{\lambda,Q}^r : E_Q^r \to D_{\lambda,Q}^r$ to be the composition of $\theta_{\lambda,Q}:E_Q^r \to D_{\lambda,Q}$ with restriction to Γ, (6.2,p.126). By (6.5f) we may factor $\theta^{\Gamma}_{\lambda,Q}$ through $E^{\Gamma}_{K}\{C(\lambda)\}$ giving a s.e.s.

6.5h

$$\hat{M}_{\lambda,Q} \to E^{\Gamma}_{Q}\{C(\lambda)\} \xrightarrow{\hat{\theta}^{\Gamma}_{\lambda,Q}} D^{\Gamma}_{\lambda,Q}$$

where $\hat{\theta}^{\Gamma}_{\lambda,Q}$ is the map induced by $\theta^{\Gamma}_{\lambda,Q}$ and $\hat{M}_{\lambda,Q} = w^{\lambda}_{Q}(M_{\lambda,Q})$ which is spanned by the set of elements $\hat{R}_3 \cup \hat{R}_4$ where $\hat{R}_4 = \{\beta^{\Gamma,+}_{ab}(e_i)\{C(\lambda)\}$: $a,b \in J_0(r)$, $i \in I(n,r)\}$.

Also, by further restriction we have the s.e.s.

6.5i

$$\hat{M}_{\lambda,Z} \to E^{\Gamma}_{Z}\{C(\lambda)\} \xrightarrow{\hat{\theta}^{\Gamma}_{\lambda,Q}} D^{\Gamma}_{\lambda,Z}$$

where $\hat{M}_{\lambda,Z} = w^{\lambda}_{Q}(M_{\lambda,Z}) = M_{\lambda,Z}\cdot\{C(\lambda)\}$.

As for (6.3c), since $D^{\Gamma}_{\lambda,Z}$ is a free Z-module, we may tensor this s.e.s. with any infinite field (char $K \neq 2$) to yield a s.e.s.

6.5j

$$K \otimes \hat{M}_{\lambda,Z} \to K \otimes E^{\Gamma}_{Z}\{C(\lambda)\} \xrightarrow{1\otimes\hat{\theta}^{\Gamma}_{\lambda,Q}} K \otimes D^{\Gamma}_{\lambda,Z} .$$

Now, $E^{\Gamma}_{K}\{C(\lambda)\}$ is naturally isomorphic to $K \otimes E^{\Gamma}_{Z}\{C(\lambda)\}$ via the $K\Gamma$ isomorphism $\hat{\alpha}: e^{K}_{i}\{C(\lambda)\} \to 1 \otimes e_i\{C(\lambda)\}$ $(i \in I(n,r),\ e^{K}_{i} \in E^{\Gamma}_{K})$ and we may therefore identify $K \otimes \hat{M}_{\lambda,Z}$ with a $K\Gamma$ submodule of $E^{\Gamma}_{K}\{C(\lambda)\}$. It is evident that identifying $K \otimes \hat{M}_{\lambda,Z}$ as a submodule of $E^{\Gamma}_{K}\{C(\lambda)\}$ is as good as identifying $K \otimes M_{\lambda,Z}$ as a submodule of E^r_K, our aim as stated in (6.3).

6.5k

For any infinite field K (char $K \neq 2$) define $\hat{M}_{\lambda,K} \subset E^r_K(C(\lambda))$ to be the $K\Gamma$ submodule $\bar{N}_{\lambda,K} + \sum_{a,b} \mathrm{Im}_K \beta^{\lambda,+}_{ab}$ (sum over all $(a,b) \in \Omega^\lambda$ = { subsets a,b of distinct columns of T^λ with $|a| = |b|$}) .

6.5ℓ Remark

When $K = Q$ (or any field of characteristic zero), by (6.4k(ii)) $\mathrm{Im}_Q \beta^{r,+}.(C(\lambda)) = \sum_{a,b} \mathrm{Im}_Q \beta^{\lambda,+}_{ab}$, therefore our definition of $\hat{M}_{\lambda,Q}$ is consistent with the definition of (6.5h). Notice also that $\hat{M}_{\lambda,Z} = \hat{M}_{\lambda,Q} \cap E^r_Z(C(\lambda))$ contains each $\beta^{\lambda,+}_{ab}(e_i(C(\lambda)))$ ($i \in I(n,r)$, a,b as above). Therefore $\bar{\alpha}(\hat{M}_{\lambda,K}) \subset K \otimes \hat{M}_{\lambda,Z} \subset K \otimes E^r_Z(C(\lambda))$.

As for (6.3e) we have the following commutative diagram:

6.5m

$$\begin{array}{ccccc}
K \otimes \hat{M}_{\lambda,Z} & \rightarrow & K \otimes E^r_Z(C(\lambda)) & \xrightarrow{1 \otimes \bar{\theta}^r_{\lambda,Q}} & K \otimes D^r_{\lambda,Z} \\
\hat{i} \uparrow & & \uparrow \bar{\alpha} & & \updownarrow \bar{j} \\
\hat{M}_{\lambda,K} & \rightarrow & E^r_K(C(\lambda)) & \xrightarrow{\bar{\theta}^r_{\lambda,K}} & D^r_{\lambda,K}
\end{array}$$

where $\hat{i}$ is the $K\Gamma$ monomorphism induced by $\bar{\alpha}$ (6.5ℓ), the $K\Gamma$ module $D^r_{\lambda,K} := E^r_K(C(\lambda))/\hat{M}_{\lambda,K}$ with $\bar{\theta}^r_{\lambda,K}$ the natural $K\Gamma$ map and $\bar{j}$ the $K\Gamma$ epimorphism given by:

$$\hat{j}(\hat{\theta}^{\Gamma}_{\lambda,K}(x)) = 1 \otimes \hat{\theta}^{\Gamma}_{\lambda,Q}(\hat{\alpha}(x)) \qquad (x \in E^{\Gamma}_{K}(C(\lambda))) \quad .$$

Our aim is to show that $\hat{j}$ is injective and therefore an isomorphism, it will then follow that $\hat{i}(\hat{M}_{\lambda,K}) = K \otimes \hat{M}_{\lambda,Z}$ and we will have our description of $K \otimes \hat{M}_{\lambda,Z}$.

6.5n Remark

From (6.5m) we have:

$$\dim_K(D^{\Gamma}_{\lambda,K}) \geq \dim_K(K \otimes D^{\Gamma}_{\lambda,Z}) = Z\text{-rank } D^{\Gamma}_{\lambda,Z} = \dim_Q D^{\Gamma}_{\lambda,Q} \quad .$$

6.5o

Recall lemma (6.4m) in which we defined γ^{rs}_{ab} , $\gamma^{rs,+}_{ab}$, the $K\Gamma_A$ maps constructed using $A : E_K \times E_K \to K$ (5.1,p.100) as β^{rs}_{ab} and $\beta^{rs,+}_{ab}$ were constructed using $B : E_K \times E_K \to K$. Similarily, we can define $K\Gamma_A$-maps γ^{λ}_{ab} , $\gamma^{\lambda,+}_{ab}$ constructed using A as β^{λ}_{ab} , $\beta^{\lambda,+}_{ab}$ were constructed using B (6.5b,c). Then (6.4m) gives:

6.5p Lemma

Let K be algebraically closed. Then there is some non singular $P : E_K \times E_K \to K$ such that

$$\operatorname{Im}_K \beta^{\lambda,+}_{ab} = P^{\lambda}.\operatorname{Im}_K \gamma^{\lambda,+}_{ab} \qquad (a,b) \in \Omega^{\lambda}$$

$$\ker_K \beta^{\lambda}_{ab} = P^{\lambda}.\ker_K \gamma^{\lambda}_{ab} \qquad (a,b) \in \Omega^{\lambda}$$

where $P^{\lambda}:E_K^r(C(\lambda)) \to E_K^r(C(\lambda))$ is the non singular map induced by $P:E_K \times E_K \to K$.

6.5q

Define $\hat{M}_{\lambda,K}$ to be the $K\Gamma_A$ submodule $\hat{N}_{\lambda,K} + \Sigma \operatorname{Im}\gamma_{ab}^{\lambda,+}$ (sum over all $(a,b) \in \Omega^{\lambda}$) . Then (6.5p) implies $P^{\lambda}.\hat{M}_{\lambda,K} = \hat{M}_{\lambda,K}$ (remember that $P \in G$ so $P^{\lambda}.\hat{N}_{\lambda,K} = \hat{N}_{\lambda,K}$) .

We state, without proof, two results of Lancaster and Towber [LT]:

6.5r Proposition [LT, Prop. 8.6]

Fix $\lambda \in \Lambda^{r,+}(G)$, λ admissible and let $f : I(n,r) \to D$ be any map with values in an abelian group D satisfying the following conditions: let $i \in I(n,r)$

(i) $f(i) = 0$, if T_i has equal entries in two distinct places of some column.

(ii) $f(i\sigma) = s(\sigma)f(i)$ any $\sigma \in C(\lambda)$.

(iii) (Garnir relations) $\sum_{\sigma\in G(J)} s(\sigma)f(i\sigma) = 0$

(for definition of $G(J)$ see (5.1m)).

(iv) For a,b subsets of the elements of distinct columns of T^{λ} with $|a| = f = |b|$ then

$$\sum_{1\le h_1<h_2<\ldots<h_f\le n} f(i(a_1,a_2,\ldots,a_f,b_1,\ldots,b_f:h_1,h_2,\ldots,h_f,h_1,\ldots,h_f) = 0 .$$

Then $\operatorname{im} f$ lies in the subgroup of D generated by the set $\{f(i) : T_i$ standard and regular$\}$ (for definition of regular, which is not used explicitly here, see [LT]).

6.5s Remark

The relations (i), (ii) and (iii) correspond to R_1, R_2 and R_3 and (iv) to the relations given by $\sum_{a,b} \operatorname{Im} \gamma_{ab}^{\lambda,+}$. Thus, (6.5r) says $\dim_K(E_K^\Gamma(C(\lambda))/\hat{M}_{\lambda,K}) \leq$ the number of regular standard λ tableau.

6.5t Theorem [LT, Theorem 8.7]

The number of regular standard λ tableau equals the dimension of $E_Q^\Gamma(C(\lambda))/\hat{M}_{\lambda,Q}$.

6.5u Remark

Both of these results are purely combinatorial and do not depend on Lancaster and Towbers framework.

We can now prove:

6.5v Theorem

The $K\Gamma$-map $\bar{j}:D^\Gamma_{\lambda,K} \to K \otimes D^\Gamma_{\lambda,Z}$ is injective and therefore

$$\bar{i} : (\bar{M}_{\lambda,K}) \cong K \otimes \bar{M}_{\lambda,Z} .$$

Proof

By applying (6.5r,s,t) we have:

$$\dim_K(E_K^r(C(\lambda))/\hat{M}_{\lambda,K}) \leq \dim_{\mathbb{C}}(E_{\mathbb{C}}^r(C(\lambda))/\hat{M}_{\lambda,\mathbb{C}}) \quad .$$

Now, assume K is algebraically closed, then by (6.5q) there are non singular maps $P_K:E_K \times E_K \to K$ and $P_{\mathbb{C}}:E_{\mathbb{C}} \times E_{\mathbb{C}} \to K$, such that $P_K^*\hat{M}_{\lambda,K} = \dot{M}_{\lambda,K}$ and $P_{\mathbb{C}}^*\hat{M}_{\lambda,\mathbb{C}} = \dot{M}_{\lambda,\mathbb{C}}$. Hence

$$\dim_K(E_K^r(C(\lambda))/\dot{M}_{\lambda,K}) \leq \dim_{\mathbb{C}}(E_{\mathbb{C}}^r(C(\lambda))/\dot{M}_{\lambda,\mathbb{C}}) \quad .$$

But $E_{\mathbb{C}}^r(C(\lambda))/\dot{M}_{\lambda,\mathbb{C}} = D^{\Gamma}_{\lambda,\mathbb{C}} \cong \mathbb{C} \otimes D^{\Gamma}_{\lambda,Z}$ (by (6.5ℓ) with $\mathbb{C}$ replacing Q). Thus, since $\dim_{\mathbb{C}} D^{\Gamma}_{\lambda,\mathbb{C}} = Z\text{-rank of } D^{\Gamma}_{\lambda,Z}$ then

$$\dim_K(D^{\Gamma}_{\lambda,K}) \leq Z\text{-rank}(D^{\Gamma}_{\lambda,Z})$$

and therefore by (6.5n)

$$\dim_K(D^{\Gamma}_{\lambda,K}) = Z\text{-rank }(D^{\Gamma}_{\lambda,Z}) = \dim_K(K \otimes D^{\Gamma}_{\lambda,Z})$$

and $\dot{j}$ must be an isomorphism of $K\Gamma$ modules. Now, $\dot{i}$ must also be an isomorphism of $K\Gamma$-modules and we are finished in the case K algebraically closed.

Now, let K be any infinite field (char $K \neq 2$) and $\overline{K}$ be its algebraic closure then we can identify E_K^r with a subset of $E_{\overline{K}}^r$ and therefore $E_K^r(C(\lambda))$ with a subset of $E_{\overline{K}}^r(C(\lambda))$ in the natural way. Now $\overline{K}.M_{\lambda,K} = M_{\lambda,\overline{K}}$ since both are spanned by the same set of elements. Hence

$$\dim_K\dot{M}_{\lambda,K} \geq \dim_{\overline{K}}\dot{M}_{\lambda,\overline{K}}$$

and therefore

$$\dim_K D^{\Gamma}_{\lambda,K} \leq \dim_{\bar{K}} D^{\Gamma}_{\lambda,\bar{K}} = Z\text{-rank}(D^{\Gamma}_{\lambda,Z}) \quad .$$

Then, by (6.5n) $\dim_K D^{\Gamma}_{\lambda,K} = Z\text{-rank}(D^{\Gamma}_{\lambda,Z})$ and the theorem follows as for the case K algebraically closed.

6.5w Conjecture

For any infinite field K (char $\neq 2$) $\hat{M}_{\lambda,K}$ equals the kernel of the composition of $\hat{\theta}_{\lambda,K} : E^{r}_{K}(C(\lambda)) \to D^{\Gamma}_{\lambda,K}$ with restriction $\psi_K : K_+[G] \to K[\Gamma]$.

6.6 The $K\Gamma$-modules $V^{\Gamma}_{\lambda,K}$

As always K is infinite of characteristic not two.

6.6a

For admissible $\lambda \in \Lambda^{r,+}(G)$ define $V^{\Gamma}_{\lambda,K}$ to be the orthogonal complement of $\hat{M}_{\lambda,K}$ in $E^{r}_{K}(C(\lambda))$ with respect to the form $\Lambda^{\lambda}B$ (6.5a). By (5.1n), (6.4e) and the definition of β^{λ}_{ab} $((a,b) \in \Omega^{\lambda})$ we have:

6.6b

$$V^{\Gamma}_{\lambda,K} := V_{\lambda,K} \cap \ker_K \beta^{\lambda}$$

where $V_{\lambda,K} \subset E^{r}_{K}(C(\lambda))$ is the Carter-Lusztig Weyl module (5.1j) and $\ker_K \beta^{\lambda} := \bigcap_{(a,b)\in\Omega^{\lambda}} \ker_K \beta^{\lambda}_{ab}$.

It follows that there is a non singular bilinear form

6.6c
$$(\ , \ ) \ : V^{\Gamma}_{\lambda,K} \times D^{\Gamma}_{\lambda,K} \to K$$

given by defining, for $x \in V^{\Gamma}_{\lambda,K}$, $y \in E^{\Gamma}_{K}(C(\lambda))$:

$$(x,\hat{\theta}^{\Gamma}_{\lambda,K}(y)) \ := \ \Lambda^{\lambda}B(x,y)$$

where $\hat{\theta}^{\Gamma}_{\lambda,K} : E^{\Gamma}_{K}(C(\lambda)) \to D^{\Gamma}_{\lambda,K}$ (6.5m).

6.6d Remark

When $K = Q$ by (6.4h)

$$\ker_Q \beta^{\lambda} = \ker_Q \beta.(C(\lambda))$$

and therefore $V^{\Gamma}_{\lambda,Q}$ as defined in (5.3g) coincides with our definition here, since if $x \in V_{\lambda,Q} \cap \ker_Q \beta$ then $x = y(C(\lambda))$ for some $y \in E^{\Gamma}_{Q}$.
But $x.(C(\lambda)) = y(C(\lambda)).|C(\lambda)| = |C(\lambda)|y(C(\lambda)) = |C(\lambda)|x$, so that $x = |C(\lambda)|^{-1} x(C(\lambda)) \in \ker_Q \beta(C(\lambda))$ and $V_{\lambda,Q} \cap \ker_Q \beta \subset V_{\lambda,Q} \cap \ker_Q \beta^{\lambda}$.
The reverse inclusion follows from the fact that
$\dim_Q(V_{\lambda,Q} \cap \ker_Q \beta^{\lambda}) = \dim_Q D^{\Gamma}_{\lambda,Q}$ (since (6.6b) is non singular) and
$\dim_Q(V_{\lambda,Q} \cap \ker_Q \beta) = \dim_Q D^{\Gamma}_{\lambda,Q}$ (since $(\ , \ )$ is non singular).

6.6e Remark

Having used the $K\Gamma$-maps $\beta^{rs}_{a,b}$, $\beta^{rs,+}_{a,b}$ to good effect in connection with modules for Γ we would like to prove the result corresponding to (4.6d) ie.

Conjecture

Let $r \geq 2$, then $\phi \in S_{r,K}(\Gamma)$ if and only if conditions (i) and (ii) of (4.6d) hold together with:

(iii) $$\phi\, \beta_{a,b}^{\rho\rho',+} = \beta_{a,b}^{\rho\rho',+}\, \phi$$

and

(iv) $$\phi\, \beta_{a,b}^{\rho\rho'} = \beta_{a,b}^{\rho\rho'}\, \phi$$

for all $\rho, \rho' \leq n$, $\rho + \rho' \leq r$, $a \subseteq \underline{\rho}$, $b \subseteq \underline{\rho}'$, $|a| = |b|$.

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Appendix A

2.4c Lemma

The element $d_\Gamma \in K[\Gamma]$ is not a zero divisor.

Unfortunately we have only been able to prove this lemma using algebraic variety theory. This is unsatisfactory in so far as this is the only departure from our rather puritanical approach to orthogonal group viewed as just a group of matrices.

Proof of (5.3c) (for algebraic variety theory see for example [H]).

Let D denote the determinant function in $K_+[G]$ and $D_\Gamma = \Psi_K(D)$, the restriction to Γ. Then $D_\Gamma^2 = 1$ in $K[\Gamma]$, since all elements of Γ are of determinant ± 1. Denote by Γ^+ (Γ^-) the subset of elements of Γ which have determinant $+1$ (-1). Then $\Gamma = \Gamma^+ \cup \Gamma^-$ and Γ^+ is of course the special orthogonal group. Both Γ^+, Γ^- are closed subvarieties of Γ and are in fact its connected components and therefore their coordinate rings $K[\Gamma^+]$ and $K[\Gamma^-]$, the restriction of functions in $K[\Gamma]$ to Γ^+ and Γ^- respectively, are integral domains.

Now, we

Claim

$$K[\Gamma] = (1 - D_\Gamma)K[\Gamma] \oplus (1+D_\Gamma)K[\Gamma] \quad \ldots \text{(A)}$$

Proof of claim

Now, $(1-D_\Gamma) + (1+D_\Gamma) = 2$ which is invertible in K, hence $K[\Gamma]$

(char K $\neq$ 2) , thus the identity of $K[\Gamma]$ is in $(1-D_\Gamma)K[\Gamma] + (1+D_\Gamma)K[\Gamma]$ and therefore

$$K[\Gamma] = (1-D_\Gamma)K[\Gamma] + (1+D_\Gamma)K[\Gamma] .$$

We need only show the sum is direct. Suppose $(1-D_\Gamma)x = (1+D_\Gamma)y$ $(x,y \in K[\Gamma])$ then multiplying through by $(1-D_\Gamma)$ we get

$$(1-D_\Gamma)^2 x = (1-D_\Gamma^2)y$$

hence

$$(1-2D_\Gamma+D_\Gamma^2)x = 0 \qquad (D_\Gamma^2 = 1)$$

and then

$$2(1-D_\Gamma)x = 0$$

so that $(1-D_\Gamma)x = 0$, proving the claim.

We can identify $(1+D_\Gamma)K[\Gamma]$ with $K[\Gamma^+]$ as follows:

let $\Psi^+ : K[\Gamma] \to K[\Gamma^+]$ be the restriction map. Clearly $(1-D_\Gamma)K[\Gamma] \subseteq \ker\Psi^+$ since $\Psi^+(D_\Gamma) = 1$ in $K[\Gamma^+]$. Suppose $(1+D_\Gamma)x \in \ker\Psi^+$, then $(1+D_\Gamma)x(g) = 0$ for all $g \in \Gamma^+$ and if $g \in \Gamma^-$ then $(1+D_\Gamma)x(g) = 0$ also since $D_\Gamma(g) = -1$. Hence $(1+D_\Gamma)x \equiv 0$ on Γ and hence $(1+D_\Gamma)x = 0$ in $K[\Gamma]$. Thus $(1-D_\Gamma)K[\Gamma] = \ker\Psi^+$ and we may identify $(1+D_\Gamma)K[\Gamma]$ with $K[\Gamma^+]$. Similarily, we may identify $K[\Gamma^-]$ with $(1-D_\Gamma)K[\Gamma]$ and this means that both are integral domains.

Now, $d_\Gamma = \frac{1}{2}(1-D_\Gamma)d_\Gamma + \frac{1}{2}(1+D_\Gamma)d_\Gamma$ with both terms non zero since $d_\Gamma(1_G) = 1$ and $d_\Gamma\ (\mathrm{diag}(1,1,\ldots,-1,\ldots 1)) = \pm 1$ (-1 in $(\ell+1)^{th}$ place). Suppose $d_\Gamma x = 0$ for some $x \in K[\Gamma]$, then $x = \frac{1}{2}(1-D_\Gamma)x + \frac{1}{2}(1+D_\Gamma)x$ and therefore

$$d_\Gamma x = \tfrac{1}{2}(1-D_\Gamma)d_\Gamma\tfrac{1}{2}(1-D_\Gamma)x + \tfrac{1}{2}(1+D_\Gamma)d_\Gamma\tfrac{1}{2}(1+D_\Gamma)x = 0$$

and therefore, since (A) is direct,

$$\tfrac{1}{4}\ (1-D_\Gamma)d_\Gamma.(1-D_\Gamma)x = 0$$

$$\tfrac{1}{4}\ (1+D_\Gamma)d_\Gamma.(1+D_\Gamma)x = 0$$

and since $(1-D_\Gamma)K[\Gamma]$ and $(1+D_\Gamma)K[\Gamma]$ are integral domains:

$$(1-D_\Gamma)x = 0 \quad , \quad (1+D_\Gamma)x = 0$$

implies $x = 0$, as required to prove d_Γ is not a zero divisor.

Appendix B

The symplectic and even orthogonal groups

In general, it seems that we could certainly carry out, with a little modification, our whole programme on the symplectic group $S_p(K)$. In particular, we can certainly prove Chevalley's theorem in the same way as for $O_{2\ell+1}(K)$ (just replace Γ_K by $S_p(K)$) and then compute a generating set for the kernel of restriction $K_+[G] \to K[Sp(K)]$. Weyl [W] has already done this in characteristic zero. When dealing with the Schur algebras $S_{r,Q}(Sp(Q))$ we would of course use 'Weyl operators' with respect to the form defining the symplectic group, this applies also to the 'generalised Weyl operators' of §6.

The even orthogonal group is more awkward. We can certainly prove Chevalley's theorem in the same way as for $O_{2\ell+1}(K)$ but when trying to use this to find the kernel of restriction $K_+[G] \to K[O_{2\ell}(K)]$ we have a problem; lemma (2.4c) does not hold for $O_{2\ell}(K)$ ie. the image of $d \in K_+[G]$ under restriction is a zero divisor in $K[O_{2\ell}(K)]$. This arises from the fact that the Big Cell in $O_{2\ell}(K)$ is not 'dense' in the algebraic variety sense, consisting only of matrices of determinant one. Perhaps a way around this is to define a 'twisted' version of the Big Cell in $O_{2\ell}(K)$, consisting of a set dense in $O^-_{2\ell}(K)$ and then using the union of both of these 'Big Cells'.

Index of principle notation

Throughout C , Q , Z , $Z_{\geq 0}$ and $Z_{>0}$ denote the complex numbers, rationals, integers, non negative integers and positive integers respectively and for a field K , char(K) denotes the characteristic of K .